ON HUMBERT MATRIX POLYNOMIALS

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ABSTRACT

The main aim of this paper is to define and study the Humbert matrix polynomials. An explicit representation, differential recurrence relations and a three-term matrix recurrence relation for the Humbert matrix polynomials are given. The hypergeometric matrix representations are established and expansion of the Humbert matrix polynomials as series of Hermite matrix polynomials are obtained.

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INTRODUCTION

Orthogonal matrix polynomials comprise an emerging field of study, with important results in both theory and applications continuing to appear in the literature and matrix differential equations appear as finite series solutions of second order in [1, 2, 3, 4, 5, 6, 7]. In [8, 9, 10, 11, 12, 13], extension to the matrix framework of the classical families of Hermite, Laguerre, Bessel, Gegenbauer, Jacobi and Chebyshev matrix polynomials have been proposed. Moreover, some properties of the Hermite matrix polynomials are given in [14, 15] and a generalized form of the Hermite matrix polynomials has been introduced and studied in [16, 17]. An extension to the framework of the classical families of Humbert polynomials have been introduced in [18, 19, 20, 21].

The primary goal of this paper is to consider a new system of matrix polynomials, namely the Humbert matrix polynomials. The structure of the paper is as follows: In Section 2 a definition of Humbert matrix polynomials are given. Some differential recurrence relations are established in Section 3. Moreover, hypergeometric matrix representations of these polynomials are given in Section 4. Section 5, we obtain the integral relation of Humbert matrix polynomials. Finally, the expansion of series and a connection between Humbert and Hermite’s matrix polynomials recently introduced in Section 6.

If $D_0$ is the complex plane cut along the negative real axis and $\log(z)$ denotes the principal logarithm of $z$ [9], then $z^\frac{1}{2}$ represents $\exp(\frac{1}{2}\log(z))$. If $A$ is a matrix with $\sigma(A) \subset D_0$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of $A$, then $A^2 = \sqrt{A} = \exp(\frac{1}{2}\log(A))$ denotes the image by $z^\frac{1}{2} = \sqrt{z} = \exp(\frac{1}{2}\log(z))$ of the matrix functional calculus acting on the matrix $A$.

If $A$ is a matrix in $C^{N \times N}$, its two-norm denoted by $\| A \|_2$ is defined by

$$\| A \|_2 = \sup_{x \neq 0} \frac{\| Ax \|_2}{\| x \|_2}$$

where for a vector $y$ in $C^N$, $\| y \|_2$ denotes the usual Euclidean norm of $y$, $\| y \|_2 = (y^T y)^\frac{1}{2}$.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z$, which are defined in an open set $\Omega$ of the complex plane, and if $A$ is a matrix in $C^{N \times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [22], it follows that

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B$ in $C^{n \times n}$ is a matrix for which $\sigma(B) \subset \Omega$ also and if $AB = BA$, then
Let \( A \) be a matrix in \( \mathbb{C}^{N \times N} \) such that
\[
\text{Re}(z) > 0, \quad \text{for all } z \in \sigma(A).
\] (1)

Throughout this study, a matrix polynomial of degree \( n \) in \( \mathbb{C}^{N \times N} \) is an expression of the form
\[
P_n(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_1 x + A_0
\]
where \( x \) is a real variable, \( A_j \) for \( 0 \leq j \leq n \) are \( N \times N \) complex matrices and \( A_n \neq 0 \).

The reciprocal gamma function denoted by \( \Gamma^{-1}(z) = \frac{1}{\Gamma(z)} \) is an entire function of the complex variable \( z \). Then for any matrix \( A \) in \( \mathbb{C}^{N \times N} \), the image of \( \Gamma^{-1}(z) \) acting on \( A \) denoted by \( \Gamma^{-1}(A) \) is a well-defined matrix. Then \( \Gamma(A) \) is invertible, its inverse coincides with \( \Gamma^{-1}(A) \) and one gets the formula
\[
\sum_{k=0}^{n} (-1)^k (A)_k (A + nI)^{-1} = \frac{1}{\Gamma(z)} \sum_{k=0}^{n} (-1)^k (A)_k (I - A)^{-1}, \quad 0 \leq k \leq n.
\] (2)

From (2), it is easy to find that
\[
\sum_{k=0}^{n} (-1)^k (A)_k (A + nI)^{-1} = \frac{1}{\Gamma(z)} \sum_{k=0}^{n} (-1)^k (A)_k (I - A)^{-1}, \quad 0 \leq k \leq n.
\] (3)

The hypergeometric matrix function \( F(A, B; C; z) \) has been given \[4\] for matrices \( A, B \) and \( C \) in \( \mathbb{C}^{N \times N} \) such that \( C + kI \) is invertible for all integer \( k \geq 0 \).

We will exploit the following relation due to
\[
(1-x)^{-d} = x F_d(A; -x) = \sum_{n=0}^{\infty} \frac{1}{n!} A_n x^n; \quad |x| < 1.
\] (6)

It has been seen by Defez and Jódar \[9\] that, for matrices \( A(k, n) \) and \( B(k, n) \) are matrices in \( \mathbb{C}^{N \times N} \) for \( n \geq 0 \), \( k \geq 0 \), the following relations are satisfied
\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n - 2k)
\] (7)

and
\[
\sum_{k=0}^{n} B(k, n) = \sum_{k=0}^{n} B(k, n - k).
\] (8)

Similarly, we can write
\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n + 2k),
\] (9)

\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n - k),
\] (10)

\[
\sum_{k=0}^{n} B(k, n) = \sum_{k=0}^{n} B(k, n + k),
\] (11)

and, in general,
\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n - mk),
\] (12)

\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n + mk),
\] (13)

\[
\sum_{k=0}^{n} A(k, n) = \sum_{k=0}^{n} A(k, n - mk)
\]

for \( m \) is a positive integer.

**HUMBERT MATRIX POLYNOMIALS**

Let \( A \) be a positive stable matrix in \( \mathbb{C}^{N \times N} \). We define the Humbert matrix polynomials by means of the relation
\[
F(x, t, A) = (1 - 3xt + t^2)^{-d} = \sum_{n=0}^{\infty} h_n(x) t^n; \quad |t| < \infty.
\] (13)
Note, for simplify, we can write \( F(x, t, A) = F \).

By using (6), (10) and (12), we have
\[
(1 - 3xt + t^3)^{-\frac{\alpha}{3}} = \sum_{n=0}^{\infty} \binom{\alpha}{n}(3x)^{-n} t^n
\]
\[
= \sum_{n=0}^{\infty} \frac{\binom{\alpha}{n}(3x)^{-n} t^n}{k!(n-k)!}
\]
\[
= \sum_{n=0}^{\infty} \frac{\binom{\alpha}{n}(3x)^{-n} t^n}{k!(n-3k)!}
\]
(14)

By equating the coefficients of \( t^n \) in (13) and (14), we obtain an explicit representation of the Humbert matrix polynomials in the form
\[
h_n^d(x) = \sum_{k=0}^{n \leq \frac{\alpha}{3}} \frac{(-1)^k(A)_{n-2k}(3x)^{n-3k}}{k!(n-3k)!}
\]
(15)

Clearly, \( h_n^d(x) \) is a matrix polynomial of degree precisely \( n \) in \( x \). For \( x = 0 \), it follows
\[
(1 + t^3)^{-\frac{\alpha}{3}} = \sum_{n=0}^{\infty} h_n^d(0)t^n.
\]

Also, by (6) one gets
\[
(1 + t^3)^{-\frac{\alpha}{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(A)_n t^{3n}.
\]

Therefore, we have
\[
h_n^d(0) = \frac{(-1)^n(A)_n}{n!}, \quad h_n^d(x)(0, A) = 0
\]
(16)

Equation (15) yields
\[
D h_n^d(x) = \sum_{k=0}^{n \leq \frac{\alpha}{3}} \frac{(-1)^k(A)_{n-2k}(3x)^{n-3k-1}}{k!(n-3k-1)!}
\]
(17)

and from (17) it follows that
\[
D h_{3n}^d(0) = 0, \quad D h_{3n+1}^d(0) = \frac{(-1)^n 3(A)_{3n+1}}{n!}
\]
(18)

The explicit representation (15) gives
\[
h_n^d(x) = \frac{(A)_{n}(3x)^n}{n!} + \prod_{k=3}^{n-3} x^k
\]
where \( \prod_{k=3}^{n-3} \) is a matrix polynomial of degree \( (n-3) \) in \( x \). Consequently, if \( D = \frac{d}{dx} \), then, it follows that
\[
D^n h_n^d(x) = 3^n(A)_n.
\]

**DIFFERENTIAL RECURSION RELATIONS**

By differentiating (13) with respect to \( x \) and \( t \) yields respectively
\[
\frac{\partial F(x, t, A)}{\partial x} = 3At(1 - 3xt + t^3)^{-1} F(x, t, A)
\]
(19)

and
\[
\frac{\partial F(x, t, A)}{\partial t} = 3A(x - t^2)(1 - 3xt + t^3)^{-1} F(x, t, A).
\]
(20)

So that the matrix function \( F(x, t, A) \) satisfies the partial matrix differential equation
\[
(x - t^2) \frac{\partial F(x, t, A)}{\partial x} - t \frac{\partial F(x, t, A)}{\partial t} = 0.
\]

Therefore by (13) we get
\[
\sum_{n=0}^{\infty} \int D h_n^d(x)t^n - \sum_{n=0}^{\infty} \int D h_n^d(x)t^{n+2} - \sum_{n=0}^{\infty} nh_n^d(x, A)t^n = 0
\]
where \( D = \frac{d}{dx} \)

Since \( D h_0^d = 0, D h_1^d = 3A \) and for \( n \geq 2 \), then we obtain the differential recurrence relation
\[
x D h_n^d(x) - nh_n^d(x) = D h_{n-2}^d(x, A).
\]
(21)

From (19) and (20) with the aid of (13), we get respectively the following
\[ 3A(1-3xt+t^3)^4(1-3xt+t^3)^{-4} = \sum_{n=1}^{\infty} D h_n^4(x)t^{n-1} \] (22)

and

\[ 3A(x-t^2)(1-3xt+t^3)^{-1}(1-3xt+t^3)^{-4} = \sum_{n=1}^{\infty} n h_n^4(x)t^{n-1}. \] (23)

Note that \( 1-2t^3 - 3(t-x-t^2) = 1-3xt+t^3 \). Thus by multiplying (22) by \( 1-2t^3 \) and (23) by \( 3t \) and subtracting (23) from (22), we obtain

\[ (1-2t^3)\sum_{n=1}^{\infty} D h_n^4(x)t^{n-1} - 3t\sum_{n=1}^{\infty} n h_n^4(x)t^{n-1} = 3A\sum_{n=0}^{\infty} h_n^4(x)t^n \]

or

\[ 3(A+nI)h_n^4(x) = Dh_n^4(x) - 2Dh_{n-2}^4(x). \] (24)

From (21) and (24), one gets

\[ 2x D h_n^4(x) = Dh_n^4(x) - (3A+nI) h_n^4(x). \] (25)

Substituting \( n-2 \) for \( n \) in (25) gives

\[ 2x^2 D h_{n-2}^4(x) = 2nh_n^4(x) + Dh_n^4(x) - (3A+(n-2)I) h_{n-2}^4(x). \]

Putting the resulting expression for \( Dh_{n-2}^4(x) \) into (21), gives

\[ 2x^2 D h_n^4(x) = 2nh_n^4(x) + Dh_n^4(x) - (3A+(n-2)I) h_{n-2}^4(x). \] (26)

From (23) to obtain the three terms recurrence relation in the form

\[ 3x(A+nI)h_n^4(x) = (n+1)h_{n-1}^4(x) + (3A+(n-2)I) h_{n-2}^4(x) \]

and put \( n-1 \) for \( n \)

\[ n h_n^4(x) = 3x(A+(n-1)I)h_{n-1}^4(x) - (3A+(n-3)I) h_{n-3}^4(x). \] (27)

Formulas (21), (24), (25), (26) and (27) are called the recurrence formulas for Humbert matrix polynomials.

Write (22) in the form

\[ 3A(1-3xt+t^3)^{-4}(1-3xt+t^3)^{-4} = \sum_{n=1}^{\infty} D h_n^4(x)t^{n-1} = \sum_{n=0}^{\infty} D h_{n+1}(x)t^n. \] (28)

By applying (13), it follows

\[ 3A\sum_{n=0}^{\infty} h_n^{4+r}(x)t^n = \sum_{n=0}^{\infty} D h_{n+1}(x)t^n. \] (29)

Identification of the coefficients of \( t^n \) in (28) and (29) yields

\[ D h_n^4(x) = 3A h_{n+1}^4(x) \]

this gives

\[ D h_n^4(x) = 3A h_{n+1}^4(x). \] (30)

Iteration (30) yields, for \( 0 \leq r \leq n \)

\[ D^r h_n^4(x) = 3^r(A) h_{n-r}^4(x). \] (31)

The first few Humbert matrix polynomials are listed here

\[ h_0^4(x) = 1, \]
\[ h_1^4(x) = 3xA, \]
\[ h_2^4(x) = \frac{1}{2}(3x)^2(A)_2, \]
\[ h_3^4(x) = \frac{1}{6}(3x)^3(A)_3 - 3xA \]

and

\[ h_4^4(x) = \frac{1}{24}(3x)^4(A)_4 - 3xA(A)_2. \]

**HYPERGEOMETRIC MATRIX REPRESENTATIONS OF HUMBERT MATRIX POLYNOMIALS**

Since we know that

\[ (A)_{n-2k} = (-1)^{2k}(A)_k[I - (A-nI)_{2k}]^{-1}; \quad 0 \leq 2k \leq n \] (32)

and using (2), (3) and (4) become

\[ \frac{(-1)^k}{(n-3k)!} I = \frac{(-n)^{3k}}{n!} I = \frac{(-nI)^{3k}}{n!} \quad ; \quad 0 \leq 3k \leq n, \] (33)

\[ (-nI)_{3k} = 3^k \left( \frac{-n}{3} I \right)^k \left( \frac{1-n}{3} I \right)_k \left( \frac{2-n}{3} I \right)_k, \] (34)
\[(I - A - nl)_{2k} = 2^{3k}(\frac{1}{2}(I - A - nl))_k(\frac{1}{2}(I - A - nl))_k \]  \hspace{1cm} \text{(35)}

and

\[(A)_{nk} = (A)_{n}(A + nl)_k = (A)_k(A + kl)_n \]  \hspace{1cm} \text{(36)}

the explicit representation (15) and using (32) and (33) becomes

\[
h^k_s(x) = \sum_{k=0}^{\infty} (-1)^k (A)_{nk} (I - A - nl)_{2k} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k! \]

\[
\begin{align*}
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (A)_{nk} (I - A - nl)_{2k} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k! \\
&= \sum_{k=0}^{\infty} \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k! \\
&= \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k!
\end{align*}

\[
\begin{align*}
&= \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k! \\
&= \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k! \\
&= \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k!
\end{align*}
\]

this gives hypergeometric matrix representation in the form

\[
h^k_s(x) = \frac{(A)_{nk} (I - A - nl)_{2k}}{k!} \left(\frac{1}{3}(-n)l_3, \frac{2-n}{3}l_3, \frac{2-n}{3}l_3, \frac{1}{2}(2 - A - nl)_k\right) (3x)^{n-k} k!
\]

where \(\frac{1}{3}(2 - A - nl)_k\) and \(\frac{1}{2}(2 - A - nl)_k\) are matrices in \(C^{N \times N}\) such that \(\frac{1}{2}(I - A - nl) + kl\) and \(\frac{1}{2}(I - A - nl) + kl\) are invertible for all integer \(k \geq 0\).

1 A generating function representations for Humbert matrix polynomials

Consider the series

\[
\sum_{n=0}^{\infty} [(A)_n] \left[(A)_n\right]^1 h^k_s(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-3k)!} (A)_{nk} (I - A - nl)_{2k} (3x)^{n-k} k! t^n.
\]

By using (12), (36) and (44) respectively

\[
\sum_{n=0}^{\infty} [(A)_n] \left[(A)_n\right]^1 h^k_s(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-3k)!} (A)_{nk} (I - A - nl)_{2k} (3x)^{n-k} k! t^n.
\]

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-3k)!} (A)_{nk} (I - A - nl)_{2k} (3x)^{n-k} k! t^n.
\]

By identification of the coefficients of \(t^n\), we will obtain the generating relation for the Humbert matrix polynomials in the form

\[
\sum_{n=0}^{\infty} [(A)_n] \left[(A)_n\right]^1 h^k_s(x) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(3x)^n}{n!} F_1(A + nl; \frac{1}{3}(A + nl), \frac{1}{3}(A + nl + I); \frac{1}{3}(A + nl + 2I); t^3) \]

\hspace{1cm} \text{(39)}
where \( A + nI, \frac{1}{3}(A + nl) \) and \( \frac{1}{3}(A + nl + I) \) are matrices in \( C^{N \times N} \) such that \( \frac{1}{3}(A + nl) + kl, \frac{1}{3}(A + nl + I) + kl \) and \( \frac{1}{3}(A + nl + 2l) + kl \) are invertible for all integer \( k \geq 0 \).

Therefore, by using (9) and (15) we find that

\[
1 = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

\[
t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(A + nl)_k}{k!} \left( \frac{1}{3}(B + nl + I)_k \right) \left( \frac{1}{3}(B + nl + 2l)_k \right) ; \quad t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(A + nl)_k}{k!} \left( \frac{1}{3}(B + nl + I)_k \right) \left( \frac{1}{3}(B + nl + 2l)_k \right) ; \quad t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

Now for \( A \) and \( B \in C^{N \times N} \) for which \( AB = BA \), then, we obtain another a generating function for the Humbert matrix polynomials. By identification of the coefficients of \( t^n \),

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(A + nl)_k}{k!} \left( \frac{1}{3}(B + nl + I)_k \right) \left( \frac{1}{3}(B + nl + 2l)_k \right) ; \quad t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(A + nl)_k}{k!} \left( \frac{1}{3}(B + nl + I)_k \right) \left( \frac{1}{3}(B + nl + 2l)_k \right) ; \quad t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

\[
\sum_{n=0}^{\infty} \frac{(-1)^n(A + nl)_k}{k!} \left( \frac{1}{3}(B + nl + I)_k \right) \left( \frac{1}{3}(B + nl + 2l)_k \right) ; \quad t^n = 0 \quad (x = 0) \quad (n \geq 0) \quad (x = 0) \quad (n \geq 0)
\]

**INTEGRAL RELATION OF HUMBERT MATRIX POLYNOMIALS**

The Hermite matrix polynomials of two variables will be exploited here to define a matrix version of Humbert polynomials.

If \( A \) is a positive stable matrix in \( C^{n \times n} \), we recall that if \( A \) satisfies the condition (2), the two-index Hermite matrix polynomials \( H_{n,m}(x, y, A) \) of two variables are defined by the generating function [17]

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n,m}(x, y, A) = \exp(xt \sqrt{m} A - yt I)
\]

and

\[
H_{n,m}(x, y, A) = m \sum_{i=0}^{\infty} \frac{(-1)^i y^i}{k!} (x \sqrt{m} A)^{i-1} I ; \quad n \geq 0
\]

than the expansion of \( x^n I \) in a series of two-index Hermite matrix polynomials of two variables has been given in [17]

\[
(x \sqrt{m} A)^n = m \sum_{i=0}^{\infty} \frac{y^i}{k!} (x \sqrt{m} A)^{i-1} I
\]

Putting \( m = 3 \) in (41), (42) and (43), we get

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_{n}(x, y, A) = \exp(xt \sqrt{3} A - yt I),
\]

\[
H_{n}(x, y, A) = 3 \sum_{i=0}^{\infty} \frac{(-1)^i y^i}{k!} (x \sqrt{3} A)^{i-1} I,
\]

\[
(x \sqrt{3} A)^n = 3 \sum_{i=0}^{\infty} \frac{y^i}{k!} (x \sqrt{3} A)^{i-1} I
\]

Suppose that \( A \) is a matrix in \( C^{N \times N} \) satisfying the condition (1). By (44) it follows that
\[
\begin{align*}
\Gamma^{-1}(A) & = \frac{1}{n!} \int_0^e e^{-t} t^{n-(n-1)/2} H_n(x \sqrt{A})^{-1} \frac{1}{t^2} A dt \\
& = \frac{1}{n!} \int_0^e e^{-t} t^{n-(n-1)/2} \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} t^{2k} dt \\
& = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} \int_0^e e^{-t} t^{n-(n-2k-1)/2} dt \\
& = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} \int_0^e e^{-t} t^{4+(n-2k-1)/2} dt. \\
& \text{(45)}
\end{align*}
\]

Since the summation in the right-hand side of the above equality is finite, then the series and the integral can be permuted. From the definition of Gamma matrix function in [23], we have
\[
\begin{align*}
\int_0^e e^{-t} t^{4+(n-2k-1)/2} dt &= \Gamma(A + (n-2k)I)
\end{align*}
\]

(46)

By using (2), we can write
\[
\begin{align*}
\Gamma^{-1}(A) & = \frac{1}{n!} \int_0^e e^{-t} t^{n-(n-1)/2} H_n(x \sqrt{A})^{-1} \frac{1}{t^2} A dt \\
& = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} \Gamma(A + (n-2k)I) \\
& = \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} \Gamma(A) \Gamma(A + (n-2k)I) \\
& = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (3x)^{n-3k}}{k!(n-3k)!} (A)^{n-2k} = h_n^x(x)
\end{align*}
\]

(47)

Hence, the Humbert matrix polynomials can be defined by
\[
h_n^x(x) = \sum_{k=0}^{n-1} \frac{(-1)^k (A)^{n-2k} (3x)^{n-3k}}{k!(n-3k)!}
\]

(48)

or by using the Hermite matrix polynomials of integral representation in the form
\[
h_n^x(x) = \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (A)^{n-2k} (3x)^{n-3k}}{k!(n-3k)!} \\
= \frac{1}{n!} \sum_{k=0}^{n-1} \frac{(-1)^k (A)^{n-2k} (3x)^{n-3k}}{k!(n-3k)!} H_n(x \sqrt{A})^{-1} \frac{1}{t^2} A dt.
\]

(49)

Finally, we will expand the Humbert matrix polynomials in series of Hermite matrix polynomials

\[\text{EXPANDING OF HUMBERT MATRIX POLYNOMIALS IN SERIES OF HERMITE MATRIX POLYNOMIALS}\]

In this section, the Humbert matrix polynomials can be expanded in series of Hermite matrix polynomials. For the sake of clarity, we recall that if \( A \) is a matrix in \( e^{x \cdot A} \) satisfies the condition (2), than let \( y = 1 \) in (44) and the expansion of \( Ix^n \) in a series of hermite matrix polynomials has been given in the form
\[
(x \sqrt{A})^n = \sum_{i=0}^{n} \frac{n!}{k!(n-3k)!} H_{n-3k}(x, A).
\]

(50)

Now, let us expand the Humbert matrix polynomials in a series of Hermite matrix polynomials. Employing (15) and (12) with the aid of (50), we consider the series
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k (A)^{n-2k} (3x)^{n-3k}}{k!(n-3k)!} t^n
\]

(51)

Since the matrix \( A \) commutes with itself, than we can write (51) in the form
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k 3^n}{k!(n-3k)!} (\sqrt{3A})^{n-2k} H_{n-2k}(x, A) t^{n-3k}
\]

(52)

Thus
\[
\sum_{n=0}^{\infty} 3^n (\sqrt{3A})^{n} h_n^x(x) t^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k 3^n}{k!(n-3k)!} (\sqrt{3A})^{n-2k} H_{n-2k}(x, A) t^{n-3k}.
\]
\[
\sum_{n=0}^{\infty} (\sqrt{3}A)^n h^t_n(x) y^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{k! s! n!} (A)_{n+k+3} H_n(x, A)y^{n+3k+3s}
\]
this, by using (8), yields
\[
\sum_{n=0}^{\infty} (\sqrt{3}A)^n h^t_n(x) y^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k}}{(k-s)! s! n!} (A)_{n+k+2s} H_n(x, A)y^{n+3k}. \tag{53}
\]
From the relation (12) and (13), we have
\[
(A+(n+k)I)_{n+k} = 2^{2s+1} \left( \frac{1}{2} (A+(n+k+1)I)_{n+k} \right) \left( \frac{1}{2} (A+(n+k)I)_{n+k} \right),
\]
and
\[
\frac{(-1)^s}{(k-s)! k!} I = \left( \frac{-kl}{k!} \right)_{0 \leq s \leq k}
\]
and using (8), then
\[
\sum_{n=0}^{\infty} \sum_{s=0}^{n} (\sqrt{3}A)^{n+k} h^t_n(x) y^n = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^{s+n} (A)_{n+k+2s}}{(k-s)! s! n!} H_n(x, A)y^{n+3k}
\]
\[
= \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^n}{k! s! n!} \left( \frac{1}{2} (A+(n+k+1)I)_{n+k} \right) \left( \frac{1}{2} (A+(n+k)I)_{n+k} \right)
\]
\[
= \sum_{n=0}^{\infty} H_n(x, A)y^{n+3k}
\]
In view of (12), one gets
\[
\sum_{n=0}^{\infty} (\sqrt{3}A)^n h^t_n(x) y^n = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^s}{k! (n-s)!} \left( \frac{1}{2} (A+(n-k+1)I)_{n+k} \right) \left( \frac{1}{2} (A+(n-k)I)_{n+k} \right)
\]
Therefore, by identification of coefficient of \( t^n \), we obtain an expansion of Humbert matrix polynomials as a series of Hermite matrix polynomials in the form
\[
h^t_n(x) = \sum_{m=0}^{\infty} \left( \frac{1}{2} (A+(n-k+1)I)_{n+k} \right) \left( \frac{1}{2} (A+(n-k)I)_{n+k} \right) y^m.
\]
where \(-kl\left( \frac{1}{2} (A+(n-k+1)I) _{n+k} \right)\) are matrices in \( C^{N \times N} \). The results of this paper are variant, significant and so it is interesting and capable to develop its study in the future.

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**REFERENCES**