PSEUDO INTEGRAL TERNARY SEMIGROUPS

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ABSTRACT

In this paper, the term, ‘pseudo integral ternary semigroup’ is introduced. It is proved that every pseudo symmetric ternary semigroup with nonempty kernel is a pseudo integral ternary semigroup. It is also proved that an ideal a of a ternary semigroup t is pseudo symmetric iff t\a is a pseudo integral ternary semigroup. If t is a pseudo integral ternary semigroup then it is proved that t is strongly archimedean, t is archimedean, t has no proper completely prime ideals, t has no proper completely semiprime ideals, t has no proper prime ideals, t has no proper semiprime ideals, every element in t is a k-potent element are equivalent. It is proved that if s is a maximal ternary subsemigroup of a pseudo integral ternary semigroup t such that S ∩ K = ∅ then t\s is a minimal prime ideal in t.


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INTRODUCTION

The theory of ternary algebraic system was introduced by lehmer [15] in 1932, but earlier such structures were studied by kasner [11] who gave the idea of n-ary algebras. Ternary semigroups are universal algebras with one associative ternary operation. Anjaneyulu.a [1], initiated the study of pseudo symmetric ideals, radicals, semipseudo symmetric ideals in and n(a)-semigroups. Giri and wazalwar [16] initiated the study of prime radicals in semigroups. Madhusudhana rao, anjaneyulu and gangadharrao ao [17], [18], [19], and [20] initiated the study of pseudo symmetric \gamma-ideals, prime \gamma-radicals and semipseudo symmetric \gamma-ideals in \gamma-semigroups and n(a)-\gamma-semigroups. Hewitt. E. And zuckerman h.s [5] studied about ternary operations in semigroups. Petrich.m [22] introduced about semigroups. Rusakova.s a [12] gave the idea of n-ary group theory. Santiago. M. L and bala ss [14] studied about ternary semigroups. Sioson.f.m [29] gave the idea of regular ternary semigroups. In this paper we introduce the notions of pseudo integral ternary semigroup and characterize pseudo integral semigroup.

2. PRELIMINARIES :

Definition 2.1 : let t be a non-empty set. Then t is said to be a ternary semigroup if there exist a mapping from t×t×t to t which maps (x₁,x₂,x₃) → xᵢₓ₂ₓ₃ satisfying the condition:

\[ [x₁ₓ₂ₓ₃]ₓ₄ₓ₅] = [x₁ [x₂ₓ₃ₓ₄]ₓ₅] = [x₁ₓ₂ [x₃ₓ₄ₓ₅]] \forall \ xᵢ \in t, 1 \leq i \leq 5.

Definition 2.2 : let t be ternary semigroup. A non empty subset s of t is said to be a ternary subsemigroup of t if abc ∈ s for all a,b,c ∈ s.

Note 2.3 : a non empty subset s of a ternary semigroup t is a ternary subsemigroup if and only if sss ⊆ s.

Definition 2.4 : an element a of a ternary semigroup t is said to be a left zero of t provided abc = a ∀ b,c ∈ t

Definition 2.5 : an element a of a ternary semigroup t is said to be a right zero of t provided abc = a ∀ b,c ∈ t

Definition 2.6 : an element a of a ternary semigroup t is said to be a two sided zero of t provided abc = bca = a ∀ b,c ∈ t

Definition 2.7 : an element a of a ternary semigroup t is said to be a zero of t provided abc = bca = a ∀ b,c ∈ t

Note 2.8 : if a is a two sided zero of a ternary semigroup t, then a is both left zero and right zero of t.

Definition 2.9 : an element a of a ternary semigroup t is said to be zero of t provided abc = bca = a ∀ b,c ∈ t.

Note 2.10 : if a is a zero of t, then a is a left zero, lateral zero and right zero of t.

Definition 2.11 : a ternary semigroup in which every element is a left zero is called a left zero ternary semigroup.

Definition 2.12 : a ternary semigroup in which every element is a lateral zero is called a lateral zero ternary semigroup.

Definition 2.13 : a ternary semigroup in which every element is a right zero is called a right zero ternary semigroup.

Definition 2.14 : a ternary semigroup with 0 in which the product of any three elements equal to 0 is called a zero ternary semigroup (or) null ternary semigroup.
Note 2.15: let \( t \) be a ternary semigroup. If \( t \) has a zero, let \( T^0 = t \) and if \( t \) does not have a zero, let \( T^0 \) be the ternary semigroup \( t \) with zero adjoined usually denoted by the symbol 0.

Definition 2.16: a nonempty subset \( a \) of a ternary semigroup \( t \) is said to be left ideal of \( t \) if \( b, c \in t, a \in a \) implies \( bca \in a \).

Note 2.17: a nonempty subset \( a \) of a ternary semigroup \( t \) is said to be a left ideal of \( t \) if and only if \( t\vartriangleleft a \).

Definition 2.18: a nonempty subset \( a \) of a ternary semigroup \( t \) is said to be a left ideal of \( t \) if \( b, c \in t, a \in a \) implies \( bca \in a \).

Note 2.19: a nonempty subset \( a \) of a ternary semigroup \( t \) is a left ideal of \( t \) if and only if \( t\vartriangleleft a \).

Definition 2.20: a nonempty subset \( a \) of a ternary semigroup \( t \) is said to be left ideal of \( t \) if \( b, c \in t, a \in a \) implies \( bca \in a \).

Note 2.21: a nonempty subset \( a \) of a ternary semigroup \( t \) is a right ideal of \( t \) if and only if \( a\vartriangleleft t \).

Definition 2.22: a non-empty subset \( a \) of a ternary semigroup \( t \) is said to be a ternary ideal of \( t \) if \( b, c \in t, a \in a \) implies \( bca \in a \).

Definition 2.23: a nonempty subset \( a \) of a ternary semigroup \( t \) is said to be a right ideal of \( t \) if and only if \( t\vartriangleleft a \).

Definition 2.24: an ideal \( a \) of a ternary semigroup \( t \) is said to be a completely prime ideal of \( t \) if \( x, y, z \in t \) and \( xyz \subseteq a \) implies either \( x \in a \text{ or } y \in a \text{ or } z \in a \).

Definition 2.25: an ideal \( a \) of a ternary semigroup \( t \) is said to be a prime ideal of \( t \) if \( x, y, z \in t \) and \( xyz \subseteq a \Rightarrow x \subseteq a \text{ or } y \subseteq a \text{ or } z \subseteq a \).

Definition 2.26: an element \( a \) of a ternary semigroup \( t \) is said to be semisimple if \( n \) is odd natural number then \( a \in \langle a >^n \text{ i.e. } a >^n = \langle a > \).

Definition 2.27: a ternary semigroup \( t \) is called semisimple ternary semigroup provided every element in \( t \) is semisimple.

Definition 2.28: an ideal \( a \) of a ternary semigroup \( t \) is said to be a completely semiprime ideal provided \( x, y \in t, x^n \subseteq a \) for some odd natural number \( n \) if \( a \).

Definition 2.29: an ideal \( a \) of a ternary semigroup \( t \) is said to be semiprime ideal provided \( x, y, z \in t, x^n \subseteq a \) for some odd natural number \( n \) implies \( x \subseteq a \).

Definition 2.30: a ternary semigroup \( t \) is said to be an archimedean ternary semigroup provided for any \( a, b \in t \) there exists an odd natural number \( n \) such that \( a^n \subseteq b \).

Definition 2.31: a ternary semigroup \( s \) is said to be a strongly archimedean ternary semigroup provided for any \( a, b \in t \), there is an odd natural number \( n \) such that \( a^n \subseteq b \).

Theorem 2.32: every strongly archimedean ternary semigroup is an archimedean ternary semigroup.

Definition 2.33: an ideal \( a \) of a ternary semigroup \( t \) is said to be a pseudo symmetric provided \( x, y, z \in t, xyz \subseteq a \) implies \( xyzt \subseteq a \) for all \( s, t \in t \).

Definition 2.34: a ternary semigroup \( t \) is said to be a pseudo symmetric provided every ideal is a pseudo symmetric ideal.

Theorem 2.35: let \( a \) be a semipseudo symmetric ideal of a ternary semigroup \( t \). Then the following are equivalent.

1) \( a \) is the intersection of all completely prime ideals of \( t \) containing \( a \).

2) \( a \) is the intersection of all minimal completely prime ideals of \( t \) containing \( a \).

3) \( a \) is the minimal completely semiprime ideal of \( t \) relative to containing \( a \).

4) \( a \) is \( \{ x \in a \mid x^n \subseteq a \text{ for some odd natural number } n \} \).

5) \( a \) is the intersection of all prime ideals of \( t \) containing \( a \).

6) \( a \) is the intersection of all minimal prime ideals of \( t \) containing \( a \).

7) \( a \) is the minimal prime ideals of \( t \) relative to containing \( a \).

8) \( a \) is \( \{ x \in a \mid x^n \subseteq a \text{ for some odd natural number } n \} \).

Theorem 2.36: if \( t \) is a semipseudo symmetric ternary semigroup, then the following are equivalent.

1) \( t \) is a strongly archimedean semigroup.

2) \( t \) is an archimedean semigroup.

3) \( t \) has no proper completely prime ideals.

4) \( t \) has no proper completely semiprime ideals.

5) \( t \) has no proper prime ideals.

6) \( t \) has no proper semiprime ideals.

Theorem 2.37: every prime ideal \( p \) minimal relative to containing a pseudo symmetric ideal \( a \) in a ternary semigroup \( t \) is completely prime.

3. PSUDO INTEGRAL TERNARY SEMIGROUPS:

Definition 3.1: a \( t \) be an ideal of a ternary semigroup \( t \). An element \( x \in t \) is said to be a a-potent provided there exists an odd natural number \( n \) such that \( x^n \subseteq a \).

Definition 3.2: a \( t \) be an ideal of a ternary semigroup \( t \). An ideal \( b \) of \( t \) is said to be a a-potent provided there exists an odd natural number \( n \) such that \( b^n \subseteq a \).

Note 3.3: if \( a \) is an ideal of a ternary semigroup \( t \), then every element of \( a \) is an a-potent element of \( t \) and \( a \) itself is an a-potent ideal of \( t \).

Definition 3.4: a \( t \) be an ideal of a ternary semigroup \( t \). An a-potent element \( x \) is said to be a nontrivial a-potent element of \( t \) if \( x \in a \).

Theorem 3.8: if \( m \) is a maximal ideal in a ternary semigroup \( t \) containing a pseudo symmetric ideal \( a \), then \( m \) contains all a-potent elements in \( t \) or \( t \) \( m \) is a singleton which is a-potent.

Proof: suppose \( m \) does not contain all a-potent elements. Let \( a \in t \) be any a-potent element and \( b \) be any element in \( t \) \( m \).

Since \( m \) is a maximal ideal, \( m \cup < a > = m \cup < b > \Rightarrow < a > = < b > \).

Since \( b \notin m \), we have \( b \in < a > \).

Let \( n \) be the least positive odd integer such that \( a^n \subseteq b \).

Since \( a \) is a pseudo symmetric ideal then \( a \) is a semipseudo symmetric ideal and hence \( a^{n+1} \subseteq a \).

Therefore \( k^2 \subseteq a \) and hence \( b \) is an a-potent element.

Thus every element in \( t \) \( m \) is a-potent.

Similarly we can show that if \( m \) is the least positive odd integer such that \( b^p \subseteq a \), then \( a^n \subseteq a \) for some odd natural number \( n \) such that \( a^n \subseteq b \).

Since \( m \) is a maximal ideal, we have \( a \leq b > \Rightarrow < a > = < b > \).

Now since \( a \) is a pseudo symmetric ideal, we have \( (abc)^p = (abc)^p \subseteq a \) for some \( s, t \in t \).

Let \( a, b, c \in t \).

Since \( m \) is a maximal ideal, we have \( < a > = b > = < c > \).

So \( b, c \in < a > \Rightarrow b = sat, c = uvw \).

Since \( a \subseteq b \) and hence \( a = sbt \) for some \( s, t \in t \).

Now since \( a \) is a pseudo symmetric ideal, we have \( (abc)^p = (abc)^p \subseteq a \) for some \( s, t \in t \).

Therefore \( k^2 \subseteq a \) and hence \( b \) is an a-potent element.

Which is not true.
In both the cases we have a contradiction. Hence \( a = b \).
Similarly we show that \( c = a \).

**Definition 3.9** : the intersection of all ideals of a ternary semigroup \( t \) is called kernel of \( t \) and it is denoted by \( k \).

**Definition 3.10** : a ternary semigroup \( t \) with nonempty kernel \( k \) is said to be a pseudo integral ternary semigroup provided \( k \) is a pseudo symmetric ideal.

**Theorem 3.11** : every pseudo symmetric ternary semigroup with nonempty kernel is a pseudo integral ternary semigroup.

**Proof**: let \( t \) be a pseudo symmetric ternary semigroup. Then every ideal of \( t \) is pseudo symmetric. Since kernel is the intersection of all ideals and hence kernel is again an ideal. So kernel is a pseudo symmetric ideal. Therefore \( t \) is a pseudo integral ternary semigroup.

**Theorem 3.12** : if \( t \) is a ternary semigroup with empty kernel then \( t^0 \) is a pseudo integral ternary semigroup.

**Proof**: since \( t \) has empty kernel, the kernel of \( t^0 \) is \( \{0\} \).

\[ \text{Suppose } abc = 0. \]
Then \( a = 0 \) or \( b = 0 \) or \( c = 0 \) and hence \( at^b \text{e}c = 0. \) Thus \( \{0\} \) is a pseudo symmetric ideal. Therefore the ternary semigroup \( t^0 \) is a pseudo integral ternary semigroup.

**Definition 3.14** : let \( a \) be any ideal in a ternary semigroup \( t \). Put \( t/a = t \{a\} \cup \{a\} \). Define ternary multiplication in \( t/a \) as follows. Let \( a, b, c \in t/a \), \( \{abc\} = a \cdot b \cdot c \) if \( a \cdot b \cdot c \in t/a \), \( \{abc\} = a \), otherwise. Then \( t/a \) is a ternary semigroup. The ternary semigroup \( t/a \) is called rees quotient (difference) ternary semigroup of \( t \) over the ideal \( a \).

**Theorem 3.15** : let \( t \) be a ternary semigroup. An ideal \( a \) in \( t \) is a pseudo symmetric ideal if the rees quotient (difference) ternary semigroup \( t/a \) is a pseudo integral ternary semigroup.

**Proof**: clearly \( a \) is the zero of the ternary semigroup \( t/a \) and hence kernel of \( t/a \) is an itself. So \( a \) is a pseudo symmetric ideal iff \( t/a \) is a pseudo integral ternary semigroup.

**Corollary 3.16** : every minimal prime ideal in a pseudo integral ternary semigroup is completely prime.

**Proof**: let \( t \) be a pseudo integral ternary semigroup. Then kernel \( k \) is pseudo symmetric ideal. Let \( p \) be a minimal prime ideal in \( t \). Clearly \( k \subseteq p \). Therefore \( p \) is a minimal ideal relative to containing a pseudo symmetric ideal \( k \) by theorem 2.37, \( p \) is completely prime.

**Notation 3.17** : (1) we denote \( n(k) = \{\text{the set of all k-potent elements in t}\} \). (2) \( m^+, p^+, q^+ \) denote respectively the intersection of all maximal ideals, completely prime ideals and prime ideals of a ternary semigroup.

**Corollary 3.17** : every prime ideal in a pseudo integral ternary semigroup \( t \) contains all k-potent elements and hence \( n(k) \subseteq q^+ \).

**Proof**: let \( p \) be a prime ideal of \( t \) and let \( x \) be a k-potent element.

Then \( x^p \in k \) for some odd natural number \( n \).
Since \( t \) is a pseudo integral ternary semigroup, \( k \) is a pseudo symmetric ideal.
\[ \text{We have } <x^p> \subseteq k \Rightarrow <x^p> \subseteq p \Rightarrow <x> \subseteq p \Rightarrow x \in p. \]
Therefore \( p \) contains all k-potent elements. \( x \in p \) for every \( p \Rightarrow x \in q^+ \).
Therefore \( n(k) \subseteq q^+ \).

**Corollary 3.18** : if \( t \) is a pseudo integral ternary semigroup then \( n(k) = Q^+ = P^* \).

**Proof**: since \( t \) is a pseudo integral ternary semigroup, \( k \) is a pseudo symmetric ideal.
By theorem 2.35, we have \( K_1 = K_2 = K_3 \). Therefore
\[ K_1 = P^*, K_2 = N_0(K), K_3 = Q^*. \]
Hence \( N_0(K) = Q^* = P^* \).

**Theorem 3.19** : let \( t \) be a pseudo integral ternary semigroup. Then the following are equivalent.
1. \( t \) is a strongly archimedian ternary semigroup.
2. \( t \) is an archimedian ternary semigroup.
3. \( t \) has no proper completely prime ideals.
4. \( t \) has no proper completely semiprime ideals.
5. \( t \) has no proper prime ideals.
6. \( t \) has no proper semiprime ideals.
7. Every element in \( t \) is a k-potent element.

**Proof**: since \( t \) is a pseudo integral ternary semigroup, \( k \) is a pseudo symmetric ideal. By theorem 2.32, (1) implies (2) is clear and also by theorem 2.36, (2) implies (3), (4), (5) and (6) are equivalent.

(5) \( \Rightarrow (7) \) : suppose \( t \) has no proper prime ideals \( \Leftrightarrow Q^* = T \Leftrightarrow N_0(K) = T \)
\( \Leftrightarrow \) every element in \( t \) is a k-potent element.

(7) \( \Rightarrow (1) \) : let \( t \) be a k-potent element. Let \( a \in t, b \in t \) \( \Rightarrow \) for some odd natural number \( n \), \( a^n e K \Rightarrow <a^n> \subseteq K < b > \)
\( \subseteq <a >^n < b > \). Therefore \( t \) is strongly archimedian.
Hence the given conditions are equivalent.

**Theorem 3.20** : let \( t \) be a pseudo integral ternary semigroup \( \Rightarrow \) the nonempty kernel \( k \) of \( t \) is a pseudo symmetric ideal. Since \( m \) is a maximal ideal in \( t \Rightarrow k \subseteq m \). Therefore \( m \) is a maximal ideal in \( t \) containing a pseudo symmetric ideal \( k \).
By theorem 3.8, \( m \) contains all k-potent elements in \( t \) or \( t/m \) is singleton which is k-potent.

**Theorem 3.21** : if \( s \) is a maximal ternary subsemigroup of a pseudo integral ternary semigroup \( t \) such that \( S \cap K = \emptyset \), then \( t/s \) is a minimal prime ideal in \( t \).

**Proof**: let \( y, z \in t/s \) and let \( s' \) be the ternary subsemigroup of \( t \) generated by \( s \cup \{y\} \cup \{z\} \).
Since \( s' \) contains \( s \) properly, we have \( S' \cap K = \emptyset \).
There exist \( x, x_2, x_s \in S \) for some odd natural number \( n \) such that \( x_1 y^{x_2} z^{x_s} \in K \). Put \( x = x_1 x_2 x_s \in S \).
Clearly \( x \in s \).
Since \( k \) is a pseudo symmetric ideal, we obtain \( (xyz)^{t+s+t+u+u+u+u+u} \in K \), by suitable insertion of some elements. Thus \( yxz \) is k-potent. Therefore for \( s, t \in t \), \( yxz \) is k-potent. If \( yxz \in k \), then since \( x_1 y^{x_2} z^{x_s} \in K \), we have \( t/y \in t/s \) for all \( s, t \in t \).
This is a contradiction. Thus \( t/y \in t/s \) for all \( s, t \in t \).
Similarly we can show that \( t/yz \in t/s \) and \( t/yz \in t/s \) for all \( s, t \in t \).
Therefore \( t/s \) is an ideal in \( t \).
Since \( s \) is a ternary subsemigroup of \( t \), \( t/s \) is a completely prime ideal and hence \( t/s \) is a prime ideal.
Now we show that \( t/s \) is a maximal ideal.
Let \( p \) be any prime ideal of \( t \) such that \( p \subseteq t/s \).
Let \( y \in t/s \). Then as above there is an element \( x, y, z, s, t \) such that \( yxz < k \)-potent for all \( s, t \in t \).
Since \( p \) is a prime ideal, either \( x \in p \) or \( y \in p \) or \( z \in p \).
Since \( x, z \in S \), we have \( x \not\in p, z \notin p \) and hence \( y \in p \). Therefore \( p = t\backslash s \). So \( t\backslash s \) is a minimal prime ideal in \( t \).

**Theorem 3.22**: let \( t \) be a pseudo integral ternary semigroup. A subset \( s \) of \( t \) is a maximal ternary subsemigroup of \( t \) with \( S \cap K = \emptyset \) iff \( p = t\backslash s \) is a minimal prime ideal of \( t \).

**Proof**: if \( s \) is a maximal ternary subsemigroup with \( S \cap K = \emptyset \), then by theorem 3.21, \( p = t\backslash s \) is a minimal prime ideal of \( t \). Conversely suppose that \( p = t\backslash s \) is a minimal prime ideal of \( t \). Since \( t \) is a pseudo integral ternary semigroup and \( p \) is a minimal prime ideal of \( t \), then corollary 3.16, \( p \) is completely prime and hence \( s \) is a ternary subsemigroup of \( t \). Since \( p \) is minimal, we have \( s \) is a minimal ternary subsemigroup of \( t \) with \( S \cap K = \emptyset \).

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