STOKES FLOW THROUGH POROUS CYLINDRICAL PARTICLE-IN-CELL ENCLOSING A SOLID CYLINDRICAL CORE

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ABSTRACT

This paper concern an axisymmetric incompressible Stokes flow past a swarm of porous cylindrical particles enclosing a solid cylindrical core, using cylinder-in-cell model. The flow has been divided into two regions. The region I is the region outside the porous cylindrical particle enclosed by a hypothetical cylindrical cell in which the flow is governed by Stokes equation. However, region II is the porous cylinder enclosing a solid cylindrical core and flow through which is governed by Brinkman equation. This model is in resemblance with solid particles having a coating of porous layers due to adsorption of polymers during the membrane filtration process. Continuity of velocity components, continuity of normal and tangential stress at the permeable boundary are used. The no slip condition at solid cylindrical core has been employed. On the hypothetical cell the uniform velocity and Mehta-Morse boundary condition is used. Some previous results for drag force have been verified. Variation of the drag coefficient with permeability parameter $\sigma$ and particle volume fraction $\gamma$ has been studied and some new results are reported. The flow patterns through both the regions have been analyzed by stream lines. Effect of particle volume fraction $\gamma$ on flow pattern is also discussed. In our opinion, these results will have significant contributions in analyzing the hydrodynamic permeability of a membrane and also in calculating the specific resistance of aggregated colloidal cake layers in membrane filtration process which is used for the wastewater treatment.

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INTRODUCTION

The problem of aquifier, oil technology, and groundwater movement among other involves the study of the behavior of flow of fluids through porous media. A porous medium usually consists of a large number of inter connected pores whose structure is usually complicated and can differ from medium to medium. Assuming that the medium has a simple structure neglecting the viscous effect in the porous medium, the flow field usually described by Darcy law. When the motion of fluid is random in nature and when the structure of the porous medium is complicated, the Darcy law is not applicable since it does not contain terms involving a velocity gradient. In order to describe the flow field in such complicated porous medium several models have come up in this direction. One such model has been developed by Saffman (1971) which is applicable to non homogeneous porous medium.

Depending on the porosity of the medium and nature of the flow, proper models are to be chosen for high porosity medium, the Brinkman model is widely applicable. The merits of Brinkman equation over Darcy equation have been stated in Ooms et al. (1970) and Neale et al. (1973) also theoretical justification for the validity of the Brinkman equation has been discussed by Howells (1974) and Saffman (1971).

In cell model technique, it is assumed that each particle is surrounded by a fluid envelope and all the disturbance due to each particle are confined to the envelope (Happel and Brenner 1983). The fluid envelope is assumed to contain the same volumetric proportion of fluid to solid as exists in the entire assemblage. Happel and Kuwabara (1974) proposed a cell model in which two concentric cylinder serve as the model for fluid moving through an assemblage of circular cylinders. These authors solved the problem when the inner cylinder is solid with respective boundary conditions on the cell surface. The Happel model assumes uniform velocity condition and no tangential stress at the cell surface, whereas the Kuwabara model assumes vanishing of vorticity in place of no tangential stress. Drag experienced by porous cylinder in a viscous fluid at low Reynolds number was evaluated by Strechkina (1979). Low Reynolds number flow past a porous spherical shell enclosing a cavity immersed in viscous fluid was studied by Iones (1973). The problem of creeping flow over a
composite sphere enclosing a solid core with porous was solved by Masliyah et al. (1987). An analytical study of the steady incompressible flow past a circular cylinder embedded in porous medium based on the Brinkman model has been reported by Pop and Cheng (1992). The problem of the steady two dimensional stokes flow stirred by an infinitesimal rotating cylinder in the annular region between two fixed concentric cylinder was tackled by Hackborn (2000). The problem of stokes flow through a swarm of circular cylinders with Happel and Kuwabara boundary condition was discussed by Deo (2004). Flow of a viscous fluid through a porous circular pipe and its surrounding porous medium and the flow around nano spheres and nano cylinder were studied by Mathews and Hill (2006). Singh and Gupta had discussed the problem of uniform flow past a permeable inhomogeneous circular cylinder by assuming that the flow in the porous cylinder is governed by Darcy law. Gupta had solved the problem of flow past a porous cylinder matched asymptotic expansion as done by Kapulam for an impervious circular cylinder. Palanippan et al. studied the two dimensional Stokes flow with permeable cylinder. Datta and Shukla calculated the Drag on the cylinder, using slip boundary condition and also deduced that the slippage on the cylinder reduces the drag. Recently a new model for calculating specific resistance of aggregated colloidal cake layers in membrane filtration process was discussed by Kim and Yuan. Fillippov used the cell model for the permeability of membranes of porous particle with a permeable shell. They studied the influence of porous shell on permeability by applying Mehta-Morse boundary condition on the cell boundary. They also studied the hydro dynamically permeability of membranes built by particle covered by porous shell using Mehta-Morse boundary condition they apply four condition Happel, Kuwabara, Kvashnin and Cunningham (usually referred as Mehta-Morse boundary condition) and compared the result.

2. Statement and mathematical formulation of the problem

Here we have considered an axi-symmetric Stokes flow of a viscous incompressible fluid through a swarm of porous cylindrical particles of radius \(b\) enclosing an impermeable core of radius \(a\). The above model is equivalent to a co-axial porous cylindrical shell enclosing the impermeable core. Further, we assume that, this porous shell is enveloped by a concentric cylinder of radius \(c > b\), named as core cell surface. The Stokes flow of a Newtonian fluid with absolute fluid viscosity is assumed to be steady and axi-symmetric. We assume that the fluid is approaching towards the cell surface as well as partially passing through the composite cylinder perpendicular to the axis of cylinder \((z\text{-axis})\) with velocity \(U\) from left to right. The radius \(c\) of hypothetical cell is chosen in such a way that the particle volume fraction \(\gamma\) of the porous cylinder is equal to the particle volume fraction of the cell, i.e. relative to this composite cylinder (i.e. a core with porous shell) in the hypothetical cell,

\[
\gamma = \frac{\pi b^2}{\pi c^2} \quad (1)
\]

**Governing equations:** The governing equations for the creeping flow of an incompressible Newtonian fluid, which lies in the region outside the porous cylindrical shell be governed by Stokes equation [Happel and Brenner [30]] as

\[
\mu_1 \nabla^2 \mathbf{v}^{(1)} = \nabla p^{(1)} \quad (2)
\]

Also, we assume that the flow inside the porous cylindrical shell is governed by the Brinkman equation [Zlatanovski [31]]

\[
\mu_2 \nabla^2 \mathbf{v}^{(2)} - \left(\frac{\mu_1}{k}\right) \mathbf{v}^{(2)} = \nabla p^{(2)} \quad (3)
\]

Here \(\mu_1\) is the viscosity of the clear fluid, \(\mu_2\) denotes the effective viscosity of porous medium, \(k\) being the permeability of porous medium. The viscosity coefficients \(\mu_1\) and \(\mu_2\) are, in general, different. Here, \(\mathbf{v}^{(i)}, p^{(i)}, i=1,2\) be the velocity vector and pressure respectively. The stream

\[
\nabla \cdot \mathbf{v}^{(i)} = 0, \quad i = 1, 2 \quad (4)
\]

These equations of continuity for axi-symmetric, incompressible viscous fluid in cylindrical polar coordinates \((r, \theta, z)\) in both regions can be written as

\[
\frac{\partial}{\partial r} \left( r v_r^{(i)} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( r v_\theta^{(i)} \right) = 0, \quad (5)
\]

where, \(v_r^{(i)}\) and \(v_\theta^{(i)}\), \(i = 1, 2\) are component of velocities in the direction of \(r\) and \(\theta\), respectively. The stream functions \(\psi^{(i)}(r, \theta)\) in both regions, satisfying equations of continuity (5) may be defined as

\[
v_r^{(i)} = \frac{1}{r} \frac{\partial \psi^{(i)}}{\partial \theta}; \quad v_\theta^{(i)} = -\frac{\partial \psi^{(i)}}{\partial r} \quad (6)
\]

Let the index in the superscript under bracket of an entity \(\chi^{(i)}, i = 1, 2\) indicates clear and porous fluid regions, respectively.

Using the following variables

\[
\psi = U b \psi^{(i)}, \quad p = \frac{\mu U}{b} p^{(i)}, \quad i = 1, 2 \quad (7)
\]

and eliminating the pressures from both equations (2) and (3) and using (4), we get the following fourth order partial differential equations, respectively as

\[
\nabla^2 (\nabla^2 - \alpha^2) \psi^{(2)} = 0, \quad (9)
\]

\[
\nabla^2 (\nabla^2 \psi^{(i)}) = 0. \quad (10)
\]

where, \(\alpha^2 = \sigma^2 / \lambda^2\) with \(\lambda^2 = \mu_1 / \mu_2, \quad \sigma^2 = b^2 / k\) and \(\nabla^2\) being the dimensionless operator defined by

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (11)
\]

The range of \(r\) and \(\theta\) in the above equations (9) and (10), with in a cylinder is given by:

\[
0 < r < \infty, \quad 0 \leq \theta \leq 2\pi. \quad (12)
\]
Furthermore, the expressions for tangential and normal stresses \( T_{ij}^{(i)} \), \( T_{rr}^{(i)} \), \( i=1,2 \) respectively are given by

\[
T_{ij}^{(i)} = \mu \left[ \frac{1}{r^2} \frac{\partial^2 \psi^{(i)}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi^{(i)}}{\partial r} - \frac{\partial^2 \psi^{(i)}}{\partial \theta^2} \right]
\]

(13)

\[
T_{rr}^{(i)} = -p^{(i)} + \frac{2\mu}{r} \left[ \frac{1}{r^2} \frac{\partial^2 \psi^{(i)}}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial \psi^{(i)}}{\partial \theta} \right]
\]

(14)

Also, the pressure may be obtained in both regions (Happel and Brenner, [30]) by integrating the following relations respectively as

\[
\frac{\partial p^{(i)}}{\partial r} = \mu \left[ \nabla^2 \psi^{(i)} - \frac{\psi^{(i)}}{r^2} - \frac{2}{r^2} \frac{\partial \psi^{(i)}}{\partial \theta} - \delta_2 \alpha^2 \psi^{(i)} \right]
\]

(15)

\[
\frac{1}{r} \frac{\partial p^{(i)}}{\partial \theta} = \mu \left[ \nabla^2 \psi^{(i)} - \frac{\psi^{(i)}}{r^2} + \frac{2}{r^2} \frac{\partial \psi^{(i)}}{\partial \theta} - \delta_2 \alpha^2 \psi^{(i)} \right]
\]

(16)

where, \( \delta_{21} = 0 \) and \( \delta_{22} = 1 \).

A suitable stream function solution of the Stokes equation (10) can be expressed as

\[
\psi^{(1)}(r, \theta) = [A_r + B_r, r + C_r, r + D_r \ln r] \sin \theta
\]

(17)

A particular solution of the Brinkman equation (9) may be written as

\[
\psi^{(2)}(r, \theta) = [A_r + B_r, r + C_r, i(\alpha \theta) + D_k \alpha(\alpha \theta)] \sin \theta
\]

(18)

Here, \( I_1(\alpha \theta) \) and \( K_1(\alpha \theta) \) are the modified Bessel's functions of the order one of the first and second kinds (Abramowitz and Stegun [32]), respectively and the dimensionless parameter \( r = r/\ell \).

3. Solution of the problem with Mehta-Morse boundary condition

The mathematically consistent boundary conditions for the concerned problem are as follows:

On the solid cylindrical core: \( (r^2 = a) \)

\[
v_r^{(2)}(r, \theta) = 0, \quad \psi^{(2)}(r, \theta) = 0.
\]

(19)

(20)

On the porous surface: \( (r^2 = b) \)

\[
v_r^{(2)}(1, \theta) = v_r^{(1)}(1, \theta), \quad \psi^{(2)}(1, \theta) = \psi^{(1)}(1, \theta),
\]

(21)

(22)

\[
T_{rr}^{(2)}(1, \theta) = T_{rr}^{(1)}(1, \theta), \quad T_{r\theta}^{(2)}(1, \theta) = T_{r\theta}^{(1)}(1, \theta)
\]

(23)

(24)

On the hypothetical cell surface:

The uniform velocity condition

\[
v_r^{(1)}(m, \theta) = U \cos \theta,
\]

(25)

And the Mehta-Morse condition on the cell surface implies that

\[
v_r^{(1)}(m, \theta) = -U \sin \theta
\]

(26)

No-slip boundary conditions are applied on the surface of the cylindrical core of radius \( a \) (equations (19) and (20)). The continuity of the normal and tangential components of velocity and stress tensor are employed at the fluid-porous interface at \( r^2 = b \) (equations (21) – (24)), and the fluid velocity \( U \), exerted by the composite cylinder is reflected in equation (25), which is also corresponding to such a solution that the cylinder moves in the same direction with velocity \( U \) where the fluid is stationary. The condition of Mehta-Morse on the cell surface, as proposed by Mehta-Morse is implemented in equation (26). Here, we have taken \( \ell = a \) and \( m = c \).

Determination of arbitrary constants:

As a result of application of the boundary conditions (19) – (26), we find that

\[
\begin{align*}
F_1, & = \frac{1}{\ell} [\alpha \ell C_1 + K(\alpha \theta) D_1].
\end{align*}
\]

(27)

\[
\begin{align*}
F_2 &= \frac{2\pi}{\ell} \left[ \frac{1}{\ell^2} \int_{0}^{1} (T_{rr}^{(1)} + T_{r\theta}^{(1)}) \cos \theta - T_{r\theta}^{(1)} \sin \theta \right] d\theta.
\end{align*}
\]

(35)

On evaluation of stress components from equations (13) and (14), one can find that

\[
T_{rr}^{(1)} = -\frac{4\mu U}{b} \left[ r B_1 + \frac{1}{m^2} C_1 - 1 - D_1 \right] \cos \theta
\]

(36)

\[
T_{r\theta}^{(1)} = -\frac{4\mu U}{b} \left[ r B_1 + \frac{1}{m^2} C_1 \right] \sin \theta
\]

(37)

Inserting the values of (36) and (37) in equation (35) and integrating, we get

\[
F = 4\pi \mu U D_1
\]

(38)
\[ \Delta = [32(1+m^4) - 4(-1+m^2)(8-\alpha^2 + m^2(8+3\alpha^2))\lambda^2 + 8(-1+m^2)\alpha^2 \lambda^2 + \alpha^3 \lambda^2 I_1(\alpha)K_1(\alpha) (4(-1+m^4) - (1-m^2)\alpha^2 \lambda^2) + (-1+m^4)\alpha^2 \lambda^2 \log m) + \alpha^2 K_2(\alpha)(2(5-8m^2 + 3m^4) - (1-m^2)\lambda^2 + (4+\alpha^2)\lambda^2) + (-1+m^4)\lambda^2 (4+\alpha^2)\lambda^2 \log m) + \alpha^2 K_2(\alpha) (-1+m^4)(8(-8+\alpha^2)\lambda^2 + (81+m^4)(8+\alpha^2)\lambda^2 \log m) + \alpha^2 (8(-1-m^4) + (1-m^4)\lambda^2) (\epsilon^2 I_1(\alpha)K_2(\alpha) - \lambda^2 \log m) + I_2(\alpha)(\alpha^2 \lambda^2) + (2(-1+m^4)\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) + K_1(\alpha)(1+8m^4) + (-1+m^4)(8+\alpha^2)\lambda^2 + (8(-1+m^4)\lambda^2 + (2(-1+m^4))\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) + \alpha^2 K_2(\alpha)(-8(1+m^4) + (-1+m^4)\lambda^2 - 2(-1+m^4)\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) \} \]

Drag force on a porous cylinder-in-cell with no impermeable core:

In a particular case of \( (\ell = 0) \) i.e. no impermeable core with in the porous cylinder \( D_1 \) reduces to \( D_H \). The drag force in this case becomes

\[ F = 4\pi \mu U D_H \]  

\[ = \frac{4\pi \mu U [\alpha^2 \{ -\alpha^4 \lambda^2 I_1(\alpha) + m^4 \alpha^4 \lambda^2 I_1(\alpha) + \alpha^3 I_2(\alpha) + 2m^4 \alpha^3 I_2(\alpha) + \alpha^3 \lambda^2 I_2(\alpha) - 2m^4 \alpha^3 \lambda^2 I_2(\alpha) \}] }{\Delta_1} \]

where,

\[ \Delta_1 = -4\alpha^3 \lambda^2 I_1(\alpha) + 4m^4 \alpha^3 \lambda^2 I_1(\alpha) - \alpha^5 \lambda^4 I_1(\alpha) + 2m^2 \alpha^5 \lambda^4 I_1(\alpha) - m^4 \alpha^5 \lambda^4 I_1(\alpha) + 8\alpha^2 I_2(\alpha) + 8m^4 \alpha^2 I_2(\alpha) + 8\alpha^2 \lambda^2 I_2(\alpha) - 8m^4 \alpha^2 \lambda^2 I_2(\alpha) + m^4 \alpha^2 \lambda^2 I_2(\alpha) + 2m^4 \alpha^2 \lambda^2 I_2(\alpha) - 4m^2 \alpha^4 \lambda^2 I_2(\alpha) + 2m^4 \alpha^4 \lambda^2 I_2(\alpha) - \alpha^5 \lambda^4 I_1(\alpha) \ln m + m^4 \alpha^5 \lambda^4 I_1(\alpha) \ln m + m^2 \alpha^4 \lambda^2 I_2(\alpha) \ln m + 2m^4 \alpha^4 \lambda^2 I_2(\alpha) \ln m \]

Drag force on a porous cylinder-in-cell with no impermeable core

If \( \beta = 0, m = 0 \) or \( \lambda = 1 \) and \( a = 0 \), i.e. \( \ell = a/b = 0 \), then cylindrical shell will reduce to a porous circular cylinder of radius \( b \). In this case, we get the value of the constant \( D_1 \) as

\[ D_1 = \frac{\alpha^2 \{ -\alpha^4 \lambda^2 I_1(\alpha) + m^4 \alpha^4 \lambda^2 I_1(\alpha) + \alpha^3 I_2(\alpha) + 4I_2(\alpha) \}}{\Delta_2} \]  

where

\[ \Delta_2 = -\alpha^4 \lambda^2 I_1(\alpha) + 4m^4 \alpha^4 \lambda^2 I_1(\alpha) - \alpha^5 \lambda^4 I_1(\alpha) + 2m^2 \alpha^5 \lambda^4 I_1(\alpha) - m^4 \alpha^5 \lambda^4 I_1(\alpha) + 8\alpha^2 I_2(\alpha) + 8m^4 \alpha^2 I_2(\alpha) + 8\alpha^2 \lambda^2 I_2(\alpha) - 8m^4 \alpha^2 \lambda^2 I_2(\alpha) + m^4 \alpha^2 \lambda^2 I_2(\alpha) + 2m^4 \alpha^2 \lambda^2 I_2(\alpha) - 4m^2 \alpha^4 \lambda^2 I_2(\alpha) + 2m^4 \alpha^4 \lambda^2 I_2(\alpha) - \alpha^5 \lambda^4 I_1(\alpha) \ln m + m^4 \alpha^5 \lambda^4 I_1(\alpha) \ln m + m^2 \alpha^4 \lambda^2 I_2(\alpha) \ln m + 2m^4 \alpha^4 \lambda^2 I_2(\alpha) \ln m \]

Thus the value of drag force from the equation (38), experienced by the porous circular cylinder in a cell, comes out as

\[ F = 4\pi \mu U \alpha^2 \{ -\alpha^4 \lambda^2 I_1(\alpha) + m^4 \alpha^4 \lambda^2 I_1(\alpha) + \alpha^3 I_2(\alpha) + 4I_2(\alpha) \} \]  

Also, the drag coefficient \( C_D \) can be written as

\[ C_D = \frac{8\pi \mu U (1+\gamma)}{2\gamma - 2 - \gamma \log \gamma - \log \gamma} \]  

5. Presentation of result and discussion:

Figure-1: Variation of Drag Coefficient \( C_D \) with particle volume fraction \( \gamma \) for various values of \( \lambda, \alpha \) and \( \ell \)

Figure-2: Variation of drag coefficient \( C_D \) with viscosity ratio \( \lambda \) for various values of \( \gamma, \alpha \) and \( \ell \)

\[ \Delta = 32(1+m^4) - 4(-1+m^2)(8-\alpha^2 + m^2(8+3\alpha^2))\lambda^2 + 8(-1+m^2)\alpha^2 \lambda^2 + \alpha^3 \lambda^2 I_1(\alpha)K_1(\alpha)(4(-1+m^4) - (1-m^2)\alpha^2 \lambda^2) + (-1+m^4)\alpha^2 \lambda^2 \log m) + \alpha^2 K_2(\alpha)(2(5-8m^2 + 3m^4) - (1-m^2)\lambda^2 + (4+\alpha^2)\lambda^2) + (-1+m^4)\lambda^2 (4+\alpha^2)\lambda^2 \log m) + \alpha^2 K_2(\alpha) (-1+m^4)(8(-8+\alpha^2)\lambda^2 + (81+m^4)(8+\alpha^2)\lambda^2 \log m) + \alpha^2 (8(-1-m^4) + (1-m^4)\lambda^2) (\epsilon^2 I_1(\alpha)K_2(\alpha) - \lambda^2 \log m) + I_2(\alpha)(\alpha^2 \lambda^2) + (2(-1+m^4)\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) + K_1(\alpha)(1+8m^4) + (-1+m^4)(8+\alpha^2)\lambda^2 + (8(-1-m^4)\lambda^2 + (2(-1+m^4))\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) + \alpha^2 K_2(\alpha)(-8(1+m^4) + (-1+m^4)\lambda^2 - 2(-1+m^4)\alpha^2 \lambda^2 + 2(1+m^4)\alpha^2 \lambda^2 \log m) \} \]
Shukla/ Stokes Flow Through Porous Cylindrical Particle-In-Cell Enclosing A Solid Cylindrical Core

**Figure-3:** Variation of Drag Coefficient $C_D$ with permeability parameter $\sigma$ for various values of $\gamma, \lambda$ and $\ell$

Stream Lines for Mehta-Morse Cell Model:

REFERENCES


