RPS-HOMEOMORPHISMS IN TOPOLOGICAL SPACES

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ABSTRACT

The authors introduced rps-closed sets and rps-open sets in topological spaces and established their relationships with some generalized sets in topological spaces. The aim of this paper is to continue the study of generalized homeomorphisms. For this we define two new classes of functions, namely rps-homeomorphisms and rps*-homeomorphisms.

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INTRODUCTION

In this paper, we introduce the concept of rps-homeomorphisms and study its relationship with homeomorphisms, g-homeomorphisms and rwg-homeomorphisms. We also introduce a new class of functions rps*-homeomorphisms which form a subclass of rps-homeomorphisms. We prove that the set of all rps*-homeomorphisms from (X,τ) onto itself is a group under the composition of functions.

2. PRELIMINARIES

Throughout this paper X and Y represent the topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a topological space X, clA and intA denote the closure of A and the interior of A in X respectively. X \ A denotes the complement of A in X. We recall the following definitions and results.

Definition 2.1
A subset A of a topological space X is called
(i) regular-open if A = int clA [10]
(ii) pre-closed if cl intA ⊆ A. [4]

Definition 2.2
A subset A of a space X is called
(i) generalized closed or g-closed if clA ⊆ U whenever A ⊆ U and U is open, [2]
(ii) regular generalized closed or rg-closed if clA ⊆ U whenever A ⊆ U and U is regular-open, [6]
The complement of a g-closed set is g-open and the complement of an rg-closed set is rg-open.

Definition 2.3 [7]
A subset A of a space X is called regular pre-semiclosed (briefly rps-closed) if spcA ⊆ U whenever A ⊆ U and U is rg-open.
The complement of an rps-closed set is rps-open.

Definition 2.4 [8] A function f: (X,τ) → (Y,σ) is called rps-irresolute if f -¹(V) is rps-closed in (X,τ) for every rps-closed set V in (Y,σ).

Definition 2.5
A function f: (X,τ) → (Y,σ) is called
(i) generalized continuous or g-continuous if f -¹(V) is g-closed in X for every closed set V in Y, [1]
(ii) rps-continuous if f -¹(V) is rps-closed in (X,τ) for every closed set V in (Y,σ) [8]

Definition 2.6 [1]
A function f: (X,τ) → (Y,σ) is called gc-irresolute if f -¹(V) is g-closed in (X,τ) for every closed set V in (Y,σ).

Definition 2.7
A bijective function f: (X,τ) → (Y,σ) is called
(i) generalized homeomorphism or g-homeomorphism if both f and f -¹ are g-continuous, [3]
(ii) regular weakly generalized homeomorphism or rwg-homeomorphism if both f and f -¹ are gc-irresolute. [5]

Definition 2.8 [9]
A function f: (X,τ) → (Y,σ) is said to be
(i) regular pre-semiclosed (briefly rps-closed) if the image of every closed set in (X,τ) is rps-closed in (Y,σ).
(ii) regular pre-semiopen (briefly rps-open) if the image of every open set in (X,τ) is rps-open in (Y,σ).

Theorem 2.9 [8]
A function f: X → Y is rps-irresolute if and only if the inverse image of every rps-closed set in Y is rps-closed in X.

Theorem 2.10 [8]
Every rps-irresolute function is rps-continuous.

3. RPS-HOMEOMORPHISMS

Definition 3.1
A bijection \( f: (X, \tau) \to (Y, \sigma) \) is called regular pre-semi homeomorphism (briefly rps-homeomorphism) if \( f \) and \( f^{-1} \) are rps-continuous. We say that the spaces \((X, \tau)\) and \((Y, \sigma)\) are rps-homeomorphic if there exists an rps-homeomorphism from \((X, \tau)\) onto \((Y, \sigma)\).

**Theorem 3.2**

Every homeomorphism is an rps-homeomorphism.

**Proof**

Let \( f: (X, \tau) \to (Y, \sigma) \) be a homeomorphism. Then \( f \) and \( f^{-1} \) are continuous and \( f \) is bijection. Since every continuous function is rps-continuous, it follows that \( f \) is rps-homeomorphism. The converse is not true as shown in the following example.

**Example 3.3**

Consider \( X = Y = \{a, b, c, d\} \) with topologies \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\} \). Let \( f(X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is rps-homeomorphism but not homeomorphism.

**Remark 3.4**

The composition of two rps-homeomorphisms need not be an rps-homeomorphism in general as seen from the following example.

**Example 3.5**

Consider \( X = Y = Z = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \} \) and \( \eta = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Z\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) be the identity functions. Then both \( f \) and \( g \) are rps-homeomorphisms, but their composition \( g \circ f: (X, \tau) \to (Z, \eta) \) is not rps-homeomorphism, because for the closed set \( \{b\} \in \{a\} \in \eta \), \( (g \circ f)^{-1}(\{b\}) = f^{-1}(g^{-1}(\{b\})) = f^{-1}(\{b\}) = \{b\} \) which is not rps-closed in \( (X, \tau) \). Therefore \( g \circ f \) is not rps-homeomorphism.

**Definition 3.6**

A bijection \( f: (X, \tau) \to (Y, \sigma) \) is said to be rps*-homeomorphism if both \( f \) and \( f^{-1} \) are rps-irresolute. We say that spaces \((X, \tau)\) and \((Y, \sigma)\) are rps*-homeomorphic if there exists an rps*-homeomorphism from \((X, \tau)\) onto \((Y, \sigma)\).

We denote the family of all rps*-homeomorphisms of a topological space \((X, \tau)\) onto itself by \( rps^*h(X, \tau) \).

**Theorem 3.7**

Every rps*-homeomorphism is an rps-homeomorphism.

**Proof**

Let \( f: (X, \tau) \to (Y, \sigma) \) be an rps*-homeomorphism. Then \( f \) and \( f^{-1} \) are rps-irresolute and \( f \) is bijection. By Theorem 2.10, \( f \) and \( f^{-1} \) are rps-continuous. Therefore \( f \) is rps-homeomorphism.

The converse is not true as shown in the following example.

**Example 3.8**

Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{b\}, X\} \) and \( \sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is rps-homeomorphism but it is not rps*-homeomorphism, because \( f \) is not rps-irresolute.

**Remark 3.9**

The concept of rps*-homeomorphism is independent with the concepts of g-homeomorphism and rwg-homeomorphism as seen from the following examples.

**Example 3.10**

Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, \{a, b, c\}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is g-homeomorphism but not rps-homeomorphism.

**Example 3.11**

Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, \{c\}, Y\} \). Define \( f: (X, \tau) \to (Y, \sigma) \) by \( f(a) = c, f(b) = b \) and \( f(c) = a \). Then \( f \) is rps-homeomorphism but not g-homeomorphism and not rwg-homeomorphism.

**Example 3.12**

Consider \( X = Y = \{a, b, c\} \) with topologies \( \tau = \{\emptyset, \{a\}, X\} \) and \( \sigma = \{\emptyset, \{a, b\}, \{a, b, \}, Y\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be the identity function. Then \( f \) is rwg-homeomorphism but not rps-homeomorphism. Thus the above discussions lead to the following implication diagram.

**Diagram 3.13**

- **Theorem 3.14**

Let \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) be rps*-homeomorphisms. Then their composition \( g \circ f: (X, \tau) \to (Z, \eta) \) is rps*-homeomorphism.

**Proof**

Suppose \( f \) and \( g \) are rps*-homeomorphisms. Since \( f \) and \( g \) are rps*-homeomorphisms, \( f \) and \( g \) are rps-irresolute. Let \( U \) be rps-open in \((Z, \eta)\). Since \( g \) is rps-irresolute, by using Theorem 2.9, \( g^{-1}(U) \) is rps-open in \((Y, \sigma)\). Since \( f \) is rps-irresolute, by using Theorem 2.9, \( f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \) is rps-open in \((X, \tau)\). Therefore \( g \circ f \) is rps-irresolute. Also for an rps-open set \( G \) in \((X, \tau)\), we have \( (g \circ f)(G) \) = \( g(f(G)) \) = \( g(W) \) where \( W = f(G) \). By hypothesis, \( f(G) \) is rps-open in \((Y, \sigma)\) and so again by hypothesis, \( g(f(G)) \) is an rps-open set in \((Z, \eta)\). That is \( (g \circ f)(G) \) is an rps-open set in \((Z, \eta)\) and therefore \( (g \circ f)^{-1} \) is rps-irresolute. Also \( g \circ f \) is a bijection. This proves \( g \circ f \) is rps*-homeomorphism.

**Theorem 3.15**

The set \( rps^*h(X, \tau) \) from \((X, \tau)\) onto itself is a group under the composition of functions.

**Proof**

Let \( f, g \in rps^*h(X, \tau) \). Then by Theorem 3.14, \( g \circ f \in rps^*h(X, \tau) \). We know that the composition of functions is associative and the identity element \( I: (X, \tau) \to (X, \tau) \) belonging to \( rps^*h(X, \tau) \) serves as the identity element. If \( f \in rps^*h(X, \tau) \) then \( f^{-1} \in rps^*h(X, \tau) \). This proves \( rps^*h(X, \tau) \) is a group under the operation of composition of functions.

**Theorem 3.16**

Let \( f: (X, \tau) \to (Y, \sigma) \) be an rps*-homeomorphism. Then \( f \) induces an isomorphism from the group \( rps^*h(X, \tau) \) onto the group \( rps^*h(Y, \sigma) \).

**Proof**

Let \( f \in rps^*h(X, \tau) \). Define a function \( \Psi: rps^*h(X, \tau) \to rps^*h(Y, \sigma) \) by \( \Psi(f)(h) = f \circ h \circ f^{-1} \) for every \( h \in rps^*h(X, \tau) \). Then \( \Psi \) is a bijection.

Further for all \( h_1, h_2 \in rps^*h(X, \tau) \), \( \Psi(f)(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \Psi(f)(h_1) \circ \Psi(f)(h_2) \).
There\ therefore $\Phi$ is a homeomorphism and so it is an isomorphism induced by $f$.

**Theorem 3.17**

Let $f: (X,\tau) \to (Y,\sigma)$ be a bijective rps-continuous function. Then the following are equivalent.

(i) $f$ is an rps-open function.
(ii) $f$ is an rps-homeomorphism.
(iii) $f$ is an rps-closed function.

**Proof**

Let $f: (X,\tau) \to (Y,\sigma)$ be a bijective rps-continuous function. Suppose (i) holds. Let $F$ be closed in $(X,\tau)$. Then $X \setminus F$ is open in $(X,\tau)$. Since $f$ is rps-open, by using Definition 2.8(ii), $f(X \setminus F)$ is rps-open in $(Y,\sigma)$. This implies $X \setminus f(F)$ is rps-open in $(Y,\sigma)$ that further implies $(f^{-1})^{-1}(F) = f(F)$ is rps-closed in $(Y,\sigma)$. Therefore $f^{-1}$ is rps-closed. This proves (i) $\Rightarrow$ (ii).

Suppose $f$ is an rps-homeomorphism. Then $f$ is bijective, $f$ and $f^{-1}$ are rps-continuous. Let $F$ be closed in $(X,\tau)$. Since $f^{-1}$ is rps-continuous, by using Definition 2.5(ii), $(f^{-1})^{-1}(F) = f(F)$ is rps-closed in $(Y,\sigma)$. Therefore by using Definition 2.8(i), $f$ is rps-closed. This proves (ii) $\Rightarrow$ (iii).

Suppose (iii) holds. Let $V$ be rps-open in $X$. Then $X \setminus V$ is rps-closed in $X$. Since $f$ is rps-closed, by using Definition 2.8(i), $f(X \setminus V)$ is rps-closed in $Y$. This implies $Y \setminus f(V)$ is rps-closed in $Y$. Therefore $f(V)$ is rps-open in $Y$. This proves (iii) $\Rightarrow$ (i).

**REFERENCES**