COMMON FIXED POINT THEOREMS FOR FOUR SELF MAPS ON A STRICT FUZZY METRIC SPACE UNDER THE INFLUENCE OF AN IMPPLICIT FUNCTION

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ABSTRACT

In this paper, we prove a common fixed point theorem for four self maps on a fuzzy metric space which are pair wise weakly compatible; satisfy common property (E.A.), under the influence of an implicit function. Under certain conditions, the result of Arihant Jain, Nirmala Gupta, and V.K. Gupta[2] follows as a corollary.

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INTRODUCTION

Arihant Jain, Nirmala Gupta, and V.K. Gupta [2] proved common fixed point theorems for semi-compatible and weakly compatible mappings in a fuzzy metric space which satisfy common property (E.A.) under the influence of an implicit function while the control function is continuous and increasing in the first argument. In this paper we obtain a variant of the above fixed point theorem by replacing semi-compatibility by weak compatibility and proving unique point of coincidence for four selfmaps, by assuming that (i) the control function is increasing in the first argument and continuous in the rest of the arguments and (ii) the fuzzy metric is strictly increasing in the time variable t.

It is evident that (i) is a restriction on the mappings while (ii) is a restriction on the space

PRELIMINARIES

We begin with some known definitions and results.

**Definition 2.1:** (Zadeh.L.A [10]) A fuzzy set A in a nonempty set X is a function with domain X and values in [0,1].

**Definition 2.2:** (Schweizer.B and Sklar. A [8]) A function *: [0,1] × [0,1] → [0,1] is said to be a continuous t-norm if * satisfies the following conditions:

For a, b, c, d ∈ [0,1]

(i) * is commutative and associative

(ii) * is continuous

(iii) a * 1 = a for all a ∈ [0,1]

(iv) a * b ≤ c * d whenever a ≤ c and b ≤ d

a * b = min(a, b), a * b = a ∨ b are examples of continuous t-norms.

**Definition 2.3:** (Kramosil. I and Michalek. J [7]) A triple (X, M, *) is said to be a fuzzy metric space (FM space, briefly) if X is a nonempty set, * is a continuous t-norm and M is a fuzzy set on X^2 × [0,∞) which satisfies the following conditions:

For x, y, z ∈ X and t > 0,

(i) M(x, y, t) > 0, M(x, y, 0) = 0

(ii) M(x, y, t) = 1 if and only if x = y

(iii) M(x, y, t) = M(y, x, t)

(iv) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s)

(v) M(x, y, t) : [0,∞) → [0,1] is left continuous.

(vi) lim_{n→∞} M(x, y, t) = 1

Then M is called a fuzzy metric space on X.

The function M(x, y, t) denotes the degree of nearness between x and y with respect to t.

**Example 2.4:** (George and Veeramani [4]) Let (X, d) be a metric space. Define a ∗ b = min{a, b} and M(x, y, t) = \frac{t}{t + d(x, y)} for all x, y ∈ X and t > 0, M(x, y, 0) = 0

Then (X, M, ∗) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

**Lemma 2.5:** (Grabiec [5]). For all x, y ∈ X, M(x, y, t) is a nondecreasing function.

**Definition 2.6:** (George. A. and Veeramani. P [4]) Let (X, M, ∗) be a fuzzy metric space. Then,

(i) A sequence \{x_n\} in X is said to be convergent to a point x ∈ X if lim_{n→∞} M(x_n, x, t) = 1 ∀ t > 0.

(ii) A sequence \{x_n\} in X is called a Cauchy sequence if lim_{n→∞} M(x_{n+p}, x_n, t) = 1 ∀ t > 0 and p = 1, 2, ...

(iii) An FM -space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.7:** Let X be a nonempty set and f and g be selfmaps on X. A point x in X is called a coincidence point of
f and g if fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

**Definition 2.8: (Jungck. G [6])** Two selfmaps S and T of a fuzzy metric space (X, M, ∗) are said to be weakly compatible if they commute at their coincidence points, that is if Sx = Tx for some x ∈ X, then STx = TSx.

**Definition 2.9: (Aamri. M and El Moutawakil. D [1])** Let A and B be maps from a fuzzy metric space (X, M, ∗) into itself. We say that the maps A and B satisfy the property (EA) if there exists a sequence (x_n) in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some z ∈ X.

**Definition 2.10: (Aamri. M and El Moutawakil. D [1])** Two pairs of selfmappings (A, S) and (B, T) defined on a fuzzy metric space (X, M, ∗) are said to be semi-compatible if there exist sequences (x_n) and (y_n) in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z \) for some z ∈ X.

**Definition 2.11: (Y.J. Cho, B.K. Sharma and D.R. Sahu [3])** Selfmaps A and S of a fuzzy metric space (X, M, ∗) are said to be semi-compatible if \( MAx_n\), Sz, t = 1 for all t > 0 whenever \( (x_n) \) is a sequence in X such that \( Ax_n, Sz_n \to 0 \), for some z ∈ X as n → ∞.

**Lemma 2.12:** Let A and B be selfmappings of a fuzzy metric space (X, M, ∗). If A and B be semi-compatible, then they are weakly compatible.

**Proof:** Suppose u is a coincidence point of A and B, so that Au = Bu.

Let \( x_n = u \) for every n in the definition of semi-compatibility.

Then \( Ax_n = Au = Au = Bu = Bu \).

Hence \( M(ABx_n, BAu, t) = 1 \).

\( \Rightarrow M(ABu, BAu, t) = 1 \).

\( \Rightarrow BAu = BAu \).

That is, A and B commute at the coincidence point u.

Hence (A, B) is weakly compatible.

**Lemma 2.13:** Let A and S be selfmappings of a fuzzy metric space (X, M, ∗) and the pair (A, S) be weakly compatible. Suppose the pair (A, S) has a unique point of coincidence. Then (A, S) has a unique common fixed point.

**Implicit Function:**

Following Singh and Jain [9], let \( φ \) be the set of all real continuous function \( φ: [0,1]^4 \to R \), nondecreasing in first argument, and satisfying the following conditions:

(i) for \( u, v \geq 0, φ(u, v, u, v) \geq 0 \) or \( φ(u, v, v, u) \geq 0 \) implies that \( u \geq v \).

(ii) \( φ(u, u, 1, 1) \) implies that \( u \geq 1 \).

Ahiraj Jain et al. [2] proved the following Theorem

**Theorem 2.14 (Ahiraj Jain et al. [2]):** Let A, B, S and T be selfmappings of a fuzzy metric space (X, M, ∗). Assume that there exist \( φ \in Ψ \) and \( k \in (0,1) \) such that

\[
φ(M(Ax, By, kr), M(Sx, Ty, r), M(Ax, Sz, t), M(By, Ty, kr)) \leq 0
\]

for all \( x, y \in X \) and \( t > 0 \).

Suppose that the pairs (A, S) and (B, T) satisfy the common property (EA), and \( S(X) \) and \( T(X) \) are closed subsets of X. Then the pairs (A, S) and (B, T) have a unique point of coincidence. Further, A, B, S and T have a unique common fixed point provided the pairs (A, S) and (B, T) are weakly compatible.

**Proof:** Since the pairs (A, S) and (B, T) satisfy the common property (EA), there exist sequences \( (x_n) \) and \( (y_n) \) in X such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z \) for some \( z \in X \).

Since \( S(X) \) is a closed subset of X,

\( \lim_{n \to \infty} Sx_n = z \in S(X) \).

Therefore, there exists a point \( u \in X \) such that \( Su = z \).

Suppose \( M(Au, z, kt) < M(Au, z, t) \), for some positive t.

Choose \( ε > 0 \) such that

\( M(Au, z, kt) + ε < M(Au, z, t) \).

Then there exists N such that

\( M(Au, By_n, kt) < M(Au, z, kt) + ε \) for \( n ≥ N \).
Now
\[\psi(M(A_u,B_v),M(S_u,T_v),M(A_u,S_u),M(B_v,T_v))\]
\[= \psi(M(A_u,B_v),M(z,z),M(A_u,S_u),M(B_v,T_v))\]
\[\leq \psi(M(A_u,z,v) + \epsilon, M(z,z,v),M(A_u,z,v),M(B_v,T_v))\] for \(n \geq N\) (since \(\psi\) is increasing in the first argument)

Letting \(n \to \infty\), since \(\psi\) is continuous in the second, third, and fourth arguments, we get
\[\psi(M(A_u,z,v) + \epsilon, M(z,z,v),M(A_u,z,v),M(B_v,T_v))\]
\[\leq \psi(M(A_u,z,v),1,M(A_u,z,v),1)\]

Thus,
\[\psi(M(A_u,z,v) + \epsilon, M(z,z,v),M(A_u,z,v),M(B_v,T_v))\]
\[\leq \psi(M(A_u,z,v),1,M(A_u,z,v),1)\]
\[\Rightarrow \psi(M(A_u,z,v),1,M(A_u,z,v),1) \geq 0\]

Hence, \(M(A_u,z,v) = 1\), consequently \(M(A_u,z,v) = 1\) for this \(t\)

(3.5.2)

Since \(M(x,y,t)\) is a strictly increasing function in \(t\), from observation 3.4, we have
\[A_u = z\]

Hence, \(A_u = S_u = z\)

Thus \(u\) is coincidence point of the pair \((A,S)\).

Since \(T(X)\) is a closed subset of \(X\),
\[z = \lim_{n \to \infty} T_y \in T(X)\]

Therefore, there exists a point \(w \in X\) such that \(T_w = z\).

Now, we prove that \(B_w = z\)

From (3.5.1) we have
\[\psi(M(A_u,B_w),M(S_u,T_w),M(A_u,S_u),M(B_w,T_w)) \geq 0\]
\[\Rightarrow \psi(M(z,B_w),M(z,z),M(B_w,z),0) \geq 0\]
\[\Rightarrow \psi(M(z,B_w),1,1,M(B_w,z)) \geq 0\]
\[\Rightarrow M(B_w,z) \geq 1\]

In view of observation 3.4, we have
\[B_w = z\]

Hence, \(B_w = T_w = z\)

Thus \(w\) is coincidence point of the pair \((B,T)\)

Therefore
\[A_u = S_u = z = B_w = T_w\]

Thus \(z\) is a point of coincidence of the pairs \((A,S)\) and \((B,T)\).

Now we prove that \(z\) is the unique point of coincidence of the pairs \((A,S)\) and \((B,T)\).

Assume that \(v\) is a point of coincidence of the pairs \((A,S)\) and \((B,T)\).

Then there exist \(p\) and \(q\) such that
\[A_p = S_p = v = B_q = T_q\]

From 3.5.1, we have
\[\psi(M(A_p,B_q),M(S_p,T_q),M(A_p,S_p),M(B_q,T_q)) \geq 0\]
\[\Rightarrow \psi(M(z,v),M(z,v),M(v,v),0) \geq 0\]
\[\Rightarrow \psi(M(z,v),1,1) \geq 0\]

Since \(\psi\) is increasing in the first argument, we have
\[\psi(M(z,v),M(z,v),1,1) \geq 0\]
\[\Rightarrow M(z,v) = 1\] for all \(t > 0\)
\[\Rightarrow z = v\]

Hence, \(z\) is the unique point of coincidence of \((A,S)\) and \((B,T)\).

Since \((A,S)\) and \((B,T)\) are weakly compatible and \(z\) is the unique point of coincidence of \((A,S)\) and \((B,T)\), by Lemma 2.13, \(z\) is the unique common fixed point of \(A,B,S\) and \(T\).

**Theorem 3.6**: Let \(A,B,S\) and \(T\) be self mappings of a strict fuzzy metric space \((X,M,\psi)\). Assume that there exist \(\psi \in \Psi\) and \(k \in (0,1)\) such that
\[\psi(M(A_x,B_y),M(S_x,T_y),M(A_x,S_x),M(B_y,T_y)) \geq 0\]
for all \(x,y \in X\) and \(t > 0\).

Suppose that the pairs \((A,S)\) and \((B,T)\) satisfy the common property \((EA)\), and \(S(x)\) and \(T(y)\) are closed subsets of \(X\).

Then the pairs \((A,S)\) and \((B,T)\) have a unique point of coincidence. Further, \(A,B,S\) and \(T\) have a unique common fixed point provided the pairs \((A,S)\) and \((B,T)\) are weakly compatible.

**Proof**: The proof is similar to that of Theorem 3.5.

**Note 3.7**: Theorem 3.5 holds even in fuzzy metric spaces where the fuzzy metric is not necessarily strict but however satisfies the following condition.
\[M(x,y,t) = 1\] for some \(t > 0\) implies \(x = y\)

(3.7.1)

If fuzzy metric satisfies (3.7.1), then from (3.5.2) follows that \(A_u = z\).

Thus we have the following theorem.

**Theorem 3.8**: Suppose \((X,M,\psi)\) is a fuzzy metric space (not necessarily strict) satisfying all the hypothesis of theorem 3.5 (or 3.6). In addition, suppose that the fuzzy metric also satisfies (3.7.1). Then the pairs \((A,S)\) and \((B,T)\) have a unique point of coincidence. Further if the pairs \((A,S)\) and \((B,T)\) are weakly compatible, then \(A,B,S\) and \(T\) have a unique common fixed point.

**Note 3.9**: It is evident that Theorem 2.14 becomes a corollary of either Theorem 3.5 or Theorem 3.6, if we assume that the fuzzy metric is a strict fuzzy metric.

**REFERENCES**

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