STRONG CONVERGENCE OF A MODIFIED MANN ITERATIVE SCHEME FOR FIXED POINT OF K-STRICLTY PSEUDO-CONTRACTIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT
In this paper, we presented and proved a strong convergence result for a new modified Mann iterative algorithm for k-strictly pseudo-contractive mappings in Hilbert spaces. Our scheme is simple and computations are made easier. This result generalises existing results in this area.

1 INTRODUCTION
Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A map $T:H\rightarrow H$ is said to be nonexpansive if for all $x,z \in H$ we have
\[ \|Tx-Tz\| \leq \|x-z\| \]
We denote the set of fixed points of $T$ by $\text{Fix}(T)$.

Definition 1.1 A map $T:H\rightarrow H$ is said to be $k$-strictly pseudo-contractive if there exists a constant $0 \leq k < 1$ such that for all $x,z \in H$
\[ \|Tx-Tz\|^2 \leq \|x-z\|^2 + k \| (I-T)x - (I-T)z \|^2 . \] (1)
In a real Hilbert space it follows that (1) is equivalent to
\[ \langle Tx-Tz, x-z \rangle \leq \|x-z\|^2 - \frac{1-k}{2} \| (I-T)x - (I-T)z \|^2 . \] (2)
Observe that the class of $k$-strictly pseudo-contractive mappings includes as a sub class the class of nonexpansive mappings i.e., when $k=0$. The mapping $T$ is as well said to be pseudo-contractive if $k=1$, and $T$ is said to be strongly pseudo-contractive if there exists $k \in (0,1)$ such that $T-kI$ is pseudo-contractive.

Iterative methods for nonexpansive mappings have been extensively studied by many authors see [3, 7, 9,17], while that of strictly pseudo-contractive maps are far less developed because the second term appearing in the right hand side of (1) posses a lot of treat in computations. However, Browder and PETryshyn in their work in 1967 initiated the study of fixed point of strictly pseudo-contractive maps. Since strictly pseudo-contractive maps is one of the most important class of mappings in nonlinear mappings, and has more interesting and powerful applications in solving inverse problems see Scherzer [13], it is of high importance to develop iterative methods for strictly pseudo-contractive mappings. However recently many authors have devoted time in developing schemes for finding fixed points for strictly pseudo-contractive maps, see [1,10] and the references therein. In 1953, for an arbitrary initial element $x_1 \in E$, Mann [6] introduced the iterative scheme
\[ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \quad \forall n \geq 1, \]
where $\{\alpha_n\}$ is a sequence in $[0,1]$. The Mann iteration has been extensively studied for nonexpansive mappings. The Mann iteration can only guarantee weak convergence in the case of infinite dimensional Hilbert Spaces (see [12, 4, 2]). However several modifications to the Mann iteration were made in recent times by many authors to obtain strong convergence results, see [11] and the references therein.

Browder and Petryshyn [3] in their work showed that for a $k$-strictly pseudo-contractive mapping $T$ such that $\text{Fix}(T) \neq \emptyset$, the sequence $\{x_n\}$ iteratively generated by arbitrary initial point $x_1 \in E$
\[ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \quad \forall n \geq 1, \]
where $\alpha$ is a constant satisfying $\lambda \leq \alpha < 1$, weakly converges to a fixed point of $T$. Recently many authors see [7] have extended Browder and Petryshyn’s result, nevertheless the convergence results they obtained are in general not strong. However, very recently, G. Marino et al [7,8], Mainge [5], G. L. Acedo [1] obtained strong convergence results for some new iterative schemes for $k$-strictly pseudo-contractive mappings.

In this paper, we consider a new modified Mann iterative scheme
\[ x_{n+1} = (1-\alpha_n) x_n + \beta_n T x_n + \alpha_n u, \quad \forall n \geq 1, \] (3)
where \( x_1 \in H \) is an arbitrary initial element. Under certain mild conditions on the sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) we prove the strong convergence of the sequence \( \{x_n\} \) generated by (3) to a fixed point of a \( k \)-strictly pseudo-contractive mapping in Hilbert spaces. The result in this paper generalizes and improves so many well known results in the literature.

### 2 Preliminaries

We present, in this section, some useful lemmas that will be used to prove our main results.

**Lemma 2.1** Let \( H \) be a real Hilbert space. Then the following inequality holds:

\[
\|x - y\|^2 = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2,
\]

for all \( x, y \in H \).

**Lemma 2.2** Let \( H \) be a real Hilbert space. Then the following inequality holds:

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,
\]

for all \( x, y \in H \).

**Lemma 2.3** \((15, 14)\) Let \( \{\alpha_n\} \) be a sequence of non negative real numbers such that

\[
\alpha_{n+1} \leq (1-\sigma_n)\alpha_n + \sigma_n \eta_n + \delta_n, \quad n \geq 1
\]

where

(i) \( \{\alpha_n\} \subset [0,1], \quad \sum_{n=1}^{\infty} \alpha_n = \infty \)

(ii) \( \limsup_{n \to \infty} \eta_n \leq 0 \)

(iii) \( \delta_n \geq 0, \quad n \geq 1, \quad \sum_{n=0}^{\infty} \delta_n < \infty \)

Then,

\[
\lim_{n \to \infty} \alpha_n = 0.
\]

**Lemma 2.4** (Demi-closed principle) \((7)\) Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T: C \to C \) be a \( k \)-strictly pseudo-contractive mapping. Then \( I - T \) is demi-closed at \( 0 \), i.e., if \( x_n \to x \in C \) and \( x_n - Tx_n \to 0 \), then \( x = Tx \).

**Lemma 2.5** \((7)\) Let \( H \) be a real Hilbert space. If \( \{x_n\} \) is a sequence in \( H \) weakly convergent to \( z \), then

\[
\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H.
\]

### 3 Main Results

**Theorem 3.1** Let \( H \) be a real Hilbert space. Let \( T: H \to H \) be a \( k \)-strictly pseudo-contractive mapping such that \( \text{Fix}(T) \neq \emptyset \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be two real sequences with \( \alpha_n \in (0, 1) \) for all \( n \geq 1 \). Assume the following conditions are satisfied:

(C1) \( \lim_{n \to \infty} \alpha_n = 0; \)

(C2) \( \sum_{n=0}^{\infty} \alpha_n = \infty; \)

(C3) \( \beta_n = \frac{1}{1 - \alpha_n}. \)

Let \( x_1 \in H \) be an initial element, then for any \( u \in H \) the sequence \( \{x_n\} \) generated by (3) converges strongly to a fixed point of \( T \).

We show first that the sequence \( \{x_n\} \) is bounded.

Let \( p \in \text{Fix}(T) \). We have

\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Tx_n - p) + \alpha_n(u - p)\|^2
\]

\[
\leq \|(1 - \alpha_n - \beta_n)(x_n - p) + \beta_n(Tx_n - p) + \alpha_n(u - p)\|^2
\]

\[
\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\beta_n(1 - \alpha_n - \beta_n) \|Tx_n - p, x_n - p\|^2
\]

\[
\leq (1 - \alpha_n - \beta_n)^2 \|x_n - p\|^2 + 2\beta_n(1 - \alpha_n - \beta_n) \|Tx_n - p\|^2 + 2\beta_n(1 - \alpha_n - \beta_n) \|x_n - p\|^2 - \frac{1 - \delta_n}{2} \|x_n - Tx_n\|^2
\]

\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + (1 - \delta_n) \|x_n - Tx_n\|^2 - (1 - \alpha_n)(1 - \beta_n) \|x_n - Tx_n\|^2
\]

\[
= (1 - \alpha_n)^2 \|x_n - p\|^2 + (1 - \delta_n) \|x_n - Tx_n\|^2
\]

\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2.
\]
Hence
\[\|\left(1 - \alpha_n - \beta_n\right)(x_n - p) + \beta_n(Tx_n - p)\| \leq (1 - \alpha_n)\|x_n - p\| \tag{9}\]
Using (8) and (9), we have that
\[
\|x_{n+1} - p\| < (1 - \alpha_n)\|x_n - p\| + \alpha_n\|u - p\|
\]
\[\leq \max\{\|x_n - p\|, \|u - p\|\}
\]
By induction, we obtain that
\[\|x_n - p\| \leq \max\{\|x_1 - p\|, \|u - p\|\}
\]
Therefore the sequence \(\{x_n\}\) is bounded.

We have that
\[
\|T^k x - p\|^2 \leq \|x - p\|^2 + \lambda\|x - Tx\|^2
\]
\[
\Rightarrow (Tx - p, Tx - p) \leq (x - p, x - Tx) + (x - p, Tx - p) + \lambda\|x - Tx\|^2
\]
\[
\Rightarrow (Tx - x, Tx - x) \leq (x - p, x - Tx) + \lambda\|x - Tx\|^2
\]
It follows that
\[\left(1 - \lambda\right)\|Tx - x\|^2 \leq 2\|x - p, x - Tx\|. \tag{10}\]
By (3), (10) and lemma (2.2), we obtain that
\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n - \beta_n)x_n + \beta_n(Tx_n + \alpha_n u - p)\|^2
\]
\[
\leq \|x_n - p\|^2 - \beta_n\|x_n - Tx_n\|^2 - 2\alpha_n\langle x_n - u, x_{n+1} - p\rangle
\]
\[
\leq \|x_n - p\|^2 - 2\beta_n\|x_n - Tx_n, x_n - p\|^2 + \beta_n\|x_n - Tx_n\|^2 - 2\alpha_n\langle x_n - u, x_{n+1} - p\rangle
\]
\[
\leq \|x_n - p\|^2 - \beta_n(1 - \lambda)\|x_n - Tx_n\|^2 - 2\alpha_n\langle x_n - u, x_{n+1} - p\rangle
\]
Since \(\{x_n\}\) is bounded, there exist a constant \(K \geq 0\) such that
\[-2\alpha_n\langle x_n - u, x_{n+1} - p\rangle \leq K \text{ for all } n \geq 1.
\]
By (11) we have
\[
\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \leq K\alpha_n
\]
We shall now show that \(\{x_n\}\) converges strongly to \(p\). To do this, we consider the following two cases:

**Case 1.** Assume that the sequence \(\|x_n - p\|\) is monotone decreasing. It follows that \(\|x_n - p\|\) converges. Immediately we obtain that
\[\|x_{n+1} - p\|^2 - \|x_n - p\|^2 \to 0, \tag{13}\]
(12), (13) and condition (C1) give that
\[\|x_n - Tx_n\| \to 0. \tag{14}\]
Define \(\omega(x) = \{x : 3x \to x\}\) i.e., the weak \(\omega\)-limit set of \(\{x_n\}\). By lemma (2.4) and (14), we obtain that \(\omega(x) \subset \text{Fix}(T)\). Hence the sequence \(\{x_n\}\) converges weakly to a fixed point \(x^*\) of \(T\).

Let \(x^*, y^* \in \omega(x)\) and \(\{x_{m_j}\}, \{x_{n_j}\}\) be subsequences of \(\{x_n\}\) such that \(x_{m_j} \to x^*\) and \(x_{n_j} \to y^*\).

We know that for any \(q \in \text{Fix}(T), \lim_{n \to \infty}\|x_n - q\|\) exists.

Therefore, by lemma (2.5), we have that
\[
\lim_{n \to \infty}\|x_n - x^*\|^2 = \lim_{j \to \infty}\|x_{m_j} - x^*\|^2
\]
\[
= \lim_{j \to \infty}\|x_{m_j} - y^*\|^2 + \|y^* - x^*\|^2
\]
\[
= \lim_{j \to \infty}\|x_{n_j} - y^*\|^2 + \|y^* - x^*\|^2
\]
\[
= \lim_{j \to \infty}\|x_{n_j} - x^*\|^2 + 2\|y^* - x^*\|^2
\]
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\[
= \lim_{n \to \infty} \| x_n - x^* \|^2 + 2 \| y^* - x^* \|^2.
\]

Hence, \( x = y^* \). Now we show that \( \{x_n\} \) converges strongly to \( x \).

We set \( z = (1 - \beta_n) x_n + \beta_n Tx_n, \forall n \geq 1 \). Therefore (3) becomes

\[
x_{n+1} = z_n - \alpha_n (x_n - u), \forall n \geq 1. \tag{15}
\]

Hence

\[
x_{n+1} = (1 - \alpha_n) z_n - \alpha_n (x_n - z_n - u)
= (1 - \alpha_n) z_n - \alpha_n [ \beta_n (x_n - T x_n) - u] \tag{16}
\]

Observe that

\[
\| z_n - x^* \|^2 = \| x_n - x^* - \beta_n (x_n - T x_n) \|^2
= \| x_n - x^* \|^2 - 2 \beta_n \langle x_n - T x_n, x_n - x^* \rangle + \beta_n^2 \| x_n - T x_n \|^2
\]

\[
\leq \| x_n - x^* \|^2 - \beta_n \| (1 - \lambda) - \beta_n \| x_n - T x_n \|^2
\]

\[
\leq \| x_n - x^* \|^2.
\]

Applying lemma 2.2 to (16), we have

\[
\| x_{n+1} - x^* \|^2 = \| (1 - \alpha_n) z_n - \alpha_n [ \beta_n (x_n - T x_n) - u] - x^* \|^2
\]

\[
\leq (1 - \alpha_n)^2 \| z_n - x^* \|^2 - 2 \alpha_n \beta_n \langle x_n - T x_n, x_{n+1} - x^* \rangle
- 2 \alpha_n \langle x^* - u, x_{n+1} - x^* \rangle
\]

\[
\leq (1 - \alpha_n) \| x_n - x^* \|^2 - 2 \alpha_n \beta_n \langle x_n - T x_n, x_{n+1} - x^* \rangle
- 2 \alpha_n \langle x^* - u, x_{n+1} - x^* \rangle. \tag{17}
\]

It is clear that

\[
\limsup_{n \to \infty} 2 \beta_n \langle x_n - T x_n, x_{n+1} - x^* \rangle - 2 \langle x^* - u, x_{n+1} \rangle \leq 0.
\]

Therefore by lemma 2.3 and (17), we obtain that \( x_n \to x^* \).

**Case 2**. Assume that the sequence \( \| x_n - p \| \) is not monotonically decreasing.

We set \( \Gamma_n = \| x_n - p \|^2 \) and let \( \tau : \mathbb{N} \to \mathbb{N} \) be a mapping defined for all \( n \geq n_0 \) (for some \( n_0 \) large enough) by

\[
\tau(n) = \max \{ k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1} \}.
\]

It is easy to see that \( \tau \) is a nondecreasing sequence satisfying \( \tau(n) \to \infty \) as \( n \to \infty \), furthermore \( \Gamma_\tau(n) \leq \Gamma_\tau(n) + 1 \) for all \( n \geq n_0 \).

By (12) we obtain that

\[
\| x_{\tau(n)} - T x_{\tau(n)} \|^2 \leq \frac{\Gamma_{\tau(n)} - \Gamma_{\tau(n) + 1} + K \alpha_{\tau(n)}}{\beta_{\tau(n)} [(1 - \lambda) - \beta_{\tau(n)}]} \to 0,
\]

thus

\[
\| x_{\tau(n)} - T x_{\tau(n)} \| \to 0.
\]

Using similar argument as in **case 1**, we immediately get that \( x_{\tau(n)} \) converges weakly to \( x^* \) as \( \tau(n) \to \infty \).

Observe that for all \( n \geq n_0 \),

\[
0 \leq \| x_{\tau(n)} - x^* \|^2 - \| x_{\tau(n)} - x^* \|^2
\]

\[
\leq \alpha_{\tau(n)} \left[ 2 \beta_{\tau(n)} \| x_{\tau(n)} - T x_{\tau(n)}, x^* - x_{\tau(n+1)} \| + 2 \| u - x^*, x_{\tau(n+1)} - x^* \| - \| x_{\tau(n)} - x^* \|^2 \right].
\]

which implies that

\[
\| x_{\tau(n)} - x^* \|^2 \leq 2 \beta_{\tau(n)} \| x_{\tau(n)} - T x_{\tau(n)}, x^* - x_{\tau(n+1)} \| + 2 \| u - x^*, x_{\tau(n+1)} - x^* \|.
\]

Hence it follows that

\[
\lim_{n \to \infty} \| x_{\tau(n)} - x^* \|^2 = 0.
\]

Hence
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\[ \lim_{n \to \infty} \tau(n) = \lim_{n \to \infty} \tau(n+1) = 0. \]

Furthermore, observe that for all \( n \geq n_0 \) if \( n \neq \tau(n) \) (i.e., \( \tau(n) < n \)) we have that

\[ \tau_n \leq \tau(n+1), \quad \text{since} \quad \tau_j < \tau(n+1) < j < n. \]

Therefore for all \( n \geq n_0 \)

\[ 0 \leq \tau_n \leq \max \{ \tau_{n+1}, \tau_{n+2}, \ldots, \tau_n \} \leq \tau(n) \rightarrow 0. \]

Hence \( \lim_{n \to \infty} \tau_n = 0 \), i.e., the sequence \( \{ \tau_n \} \) converges strongly to \( x^* \). The proof is complete.

Using Theorem 3.1, we obtain the following corollary:

**Corollary 2** Let \( H \) be a real Hilbert space. Let \( T : H \to H \) be a nonexpansive mapping such that \( \text{Fix}(T) \neq \emptyset \). Let \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) be two real sequences with \( \alpha_n \in (0,1) \) for all \( n \geq 1 \). Assume the following conditions are satisfied:

\begin{align*}
(C1) & \quad \lim_{n \to \infty} \alpha_n = 0; \\
(C2) & \quad \sum_{n=0}^{\infty} \alpha_n = \infty; \\
(C3) & \quad \beta_n = \frac{1}{1- \alpha_n}.
\end{align*}

Let \( x_0 \in H \) be an initial element, then for any \( u \in H \) the sequence \( \{ x_n \} \) generated by (3) converges strongly to a fixed point of \( T \).

**REFERENCES**


