**CON-S-NORMAL SOLUTIONS TO THE MATRIX EQUATIONS** \( K\overline{K} = I \) AND \( K\overline{K} = -I \)

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**ARTICLE INFO**

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**Key Words:** Secondary similarity of matrices, con-s-similarity of matrices, s-unitary congruence, s-normal matrix, con-s-normal matrix.

**INTRODUCTION**

The matrix equation
\[ K\overline{K} = \lambda I_n, \quad \lambda \in \mathbb{R}, \tag{1} \]
is important in the theory of con-s-normal. Recall that a con-s-similarity transforms \( A \in M_n(C) \) according to the rule
\[ A \rightarrow CA^C, \tag{2} \]
where \( C \in M_n(C) \) is an arbitrary nonsingular matrix (see [1, Section 4.6]).
Dividing both sides in (1) by \(|\lambda|\) and changing the variable, we can thereafter examine the equation
\[ K\overline{K} = I_n \tag{3} \]
or
\[ K\overline{K} = -I_n. \tag{4} \]
According to [1, Lemma 4.6.9], \( K \in M_n(C) \) is a solution to (3) if and only if \( K = CC^{-1} \).

For a nonsingular matrix \( C \), we give an example of a situation where the knowledge of this fact makes it possible to obtain significant computational advantages.

Along with \( K \), consider the class \( \Gamma_k \) of matrices that concommute with \( K \), that is, the class of matrices \( A \) such that
\[ \overline{A}K = KA \tag{6} \]
For instance, if
\[
K = \begin{pmatrix}
 1 & & \\
  & \ddots & \\
  & & 1
\end{pmatrix},
\]
then \( \Gamma_k \) is the class of centro secondary hermitian matrices. The latter can also be characterized by the scalar relations
\[ a_{ij} = \overline{a_{n-i+1,n-j+1}}, \quad i, j = 1, 2, \ldots, n. \]

Suppose that the matrix \( C \) in representation (5) is known. Then, an application of the s-similarity transformation
\[ A \rightarrow B = C^{-1}AC \tag{7} \]
to each matrix \( A \in \Gamma_k \) makes \( A \) a real matrix. We emphasize that the conversion of the entire class \( \Gamma_k \) into real matrices is attained by using a single transformation matrix \( \overline{C} \). It is clear that the solution of linear algebra problems with real matrices \( B \) requires considerably less arithmetic work than an analogous task for the original complex matrices \( B \). The gain in the computational effort is especially significant if a large series of problems with matrices in \( \Gamma_k \) has to be solved and transformation (7) can be realized by simple calculations. This is exactly the case with centrosecondaryhermitian matrices. Our goal in this paper is to describe all the solutions to Eq. (3) and (4) under one of the following additional assumptions: (a) \( K \) is an s-normal matrix; (b) \( K \) is a con-s-normal matrix.

Recall that \( A \in M_n(C) \) is called a con-s-normal matrix if
\[ AA^\theta = \overline{A^\theta A} \]
This matrix class plays the same role in the theory of s-unitary congruence’s as the conventional s-normal matrices do with respect to s-unitary similarities. Note that s-unitary congruences are a particular case of con-s-similarity transformations (2) that corresponds to s-unitary transformation matrices \( C \).
In Section 2, we list some facts related to the con-s-eigenvalues of a complex matrix. The use of these facts allows us to easily obtain descriptions of con s-normal solutions to Eqs. (3) and (4). These descriptions are given by the following two theorems:

**Theorem 1.** A con-s-normal matrix \( K \in \mathbb{M}_n (C) \) is a solution to Eq.(3) if and only if \( K = \mathbb{S} \- \mathbb{U} \) is a s-unitary s-symmetric matrix.

**Theorem 2.** A con-s-normal matrix \( K \in \mathbb{M}_n (C) \) is a solution to Eq.(4) if and only if \( K \) is a s-unitary skew-symmetric matrix.

A description of s-normal solutions to Eqs.(3) and (4) is given in Section 4. It is shown that each solution can be transformed by a real s-orthogonal s-similarity transformation into a direct sum of 1-by-1 and 2-by-2 blocks. (Only 2-by-2 blocks are present for a solution to Eq.(4).) The former are the s-eigenvalues of K, and their module are necessarily equal to unity. The latter are matrices of complex hyperbolic rotations or matrices corresponding to the products of such rotations with reflections. Each 2-by-2 block is associated with a pair of s-eigenvalues of K that obey the relation

\[
\lambda_1 \lambda_2 = 1 \quad \text{(9)}
\]

In the case of Eq.(3) and the relation

\[
\lambda_1 \lambda_2 = -1 \quad \text{(10)}
\]

In the case of Eq.(4). Moreover, for Eq.(3), it is also possible to have 2-by-2 blocks that are s-unitary s-symmetric matrices.

**2. CON-S-EIGENVALUES**

With each matrix \( A \in \mathbb{M}_n (C) \), there are associated n scalar invariants of con-s-similarity transformations, which are called the con-s-eigenvalues of A. Recall their definition as given in [3].

With a matrix \( A \in \mathbb{M}_n (C) \), we associate the matrices

\[
\lambda_i = \overline{A}A \quad \text{and} \quad A_i = AA_i
\]

Although, in general, the products AB and BA need not be s-similar, is always s-similar to \( \overline{A}A \). Therefore, in the subsequent discussion of the secondary spectral-properties of these matrices, it suffices to consider only one of them, say, \( A_i \).

The secondary spectrum of \( A_i \) has two remarkable properties:

1. It is s-symmetric about the real axis. Moreover, the s-eigenvalues \( \lambda \) and \( \overline{\lambda} \) have the same multiplicity.
2. The negative s-eigenvalues of \( A_i \) (if any) are necessarily of even algebraic multiplicity.

The proofs of these assertions can be found in [1].

Let \( \lambda_i (A_i) = [\lambda_1, ..., \lambda_n] \) be the secondary spectrum of \( A_i \).

**Definition.** The con-s-eigenvalues of A are the n scalars \( \delta_1, \delta_2, ..., \delta_n \) obtained as follows:

(a) If \( \lambda \in \lambda_i (A_i) \) does not lie on the negative real semi-axis, then the corresponding con-s-eigenvalues \( \delta_i \) is defined as the square root of \( \lambda \) with a nonnegative real part:

\[
\delta_i = \sqrt{\lambda}, \quad \text{Re} \delta_i \geq 0.
\]

The multiplicity of \( \delta_i \) is set equal to that of \( \lambda_i \).

(b) With a real negative \( \lambda \in \lambda_i (A_i) \), we associate two conjugate purely imaginary con-s-eigenvalues

\[
\delta_i = \pm \sqrt{\lambda}. \quad \text{The multiplicity of each con-s-eigenvalue is set equal to half the multiplicity of} \lambda_i.
\]

For an s-symmetric A, we have

\[
\overline{A} = A^\theta, \quad A_i = A_i^\theta.
\]

Thus, the con-s-eigenvalues of A are identical to its singular values. This reveal the following more general result:

**Theorem 3.** The singular values of a con-s-normal matrix \( A \in \mathbb{M}_n (C) \) are the moduli of its con-s-eigenvalues.

Proved in [4], this assertion is the congruent analogue of a well-known property of s-normal matrices, namely, the singular values of s-normal matrix are the moduli of its s-singular values.

**3. CON-S-NORMAL SOLUTIONS**

**3.1. Equation (3)**

Equation (3) implies that

\[
K = K = I \quad \text{(11)}
\]

that is, all the con-s-eigenvalues of K are equal to unity. By assumption, K is a con-s-normal matrix; hence, by Theorem 3, all the singular values of K are equal to unity. Thus, K is a s-unitary matrix; moreover, the relations \( KK^\theta = I \) and \( KK^\theta = I \) imply that \( K = K^5 \). Consequently, every solution to the system of matrix equations

\[
K = I, \quad KK^\theta = K^\theta K \quad \text{(12)}
\]

is an s-unitary s-symmetric matrix. The converse assertion (that is, every s-unitary s-symmetric matrix satisfies system (12)) is obvious. Theorem 1 is proved.

Let us see what can be said of representation (5) under the hypotheses of Theorem 1. We look for a s-unitary s-symmetric matrix C for this representation. Then, \( C = C^\theta \) satisifies system (13). Thus, C can be chosen as the square root of K, which is a polynomial in this matrix.

**3.2. Equation (4)**

Up to obvious modifications, the analysis given in the preceding subsection remains valid for Eq.(4). Instead of (11), we now have the relations

\[
K_i = K_i = -I. \quad \text{(13)}
\]

They mean that the con-s-eigenvalues of K are the scalars i and -i. As before, all the singular values of K are equal to unity; that is, K is an s-unitary matrix. Moreover, the relations \( KK^\theta = -I \) and \( KK^\theta = -I \) imply that \( K = -K^5 \). Thus, every solution to the system of matrix equations

\[
K = -I, \quad KK^\theta = K^\theta K \quad \text{(14)}
\]

is an s-unitary skew-s-symmetric matrix. The converse assertion (that is, every s-unitary skew-s-symmetric matrix satisfies system (13)) is obvious. Theorem 2 is proved.

**4. S-NORMAL SOLUTIONS**

**4.1. Equation (3)**

For n=1 and n=2. For n=1, Eq.(3) takes the form
That is, it describes all the complex scalars with the modulus one.

Assume that $n=2$. Let us find the solutions to the system of matrix equations.

$$ K\overline{K} = I, \quad KK^o = K^o K $$

(14)

Since $\overline{K} = K^{-1}$, the second equation in this system can be rewritten in the equivalent (and simpler) form

$$ K^S K = KK^S $$

(15)

We set

$$ K = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) $$

(16)

Then, Eq.(15) is equivalent to the three scalar equalities

$$ \alpha^2 + \beta^2 = \gamma^2 + \delta^2 = 0, \quad \alpha \beta = \gamma \delta = 0 $$

(17)

It follows from (17a)[or(17c)] that $\gamma = \pm \beta$.

These two possibilities are analyzed separately.

**Case 1:** $\gamma = \beta$. All the relations in system (17) are satisfied. Matrix (16) is s-symmetric; therefore, the first equation in (14) can be written as $KK^o = I$. Thus, $K$ is s-unitary matrix. It is easy to show that all the s-unitary s-symmetric matrices of order two are given by the formula.

$$ k = e^{i\phi} \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix}, \quad \phi, t \in [0, 2\pi]. $$

**Case 2:** $\gamma = -\beta$. Assume that $\beta \neq 0$, since the opposite situation is covered by the analysis in the preceding paragraph. In this case, (17b) reduces to the equality $\alpha = \delta$.

For the matrix

$$ K = \left( \begin{array}{cc} \alpha & \beta \\ -\beta & \alpha \end{array} \right) $$

(18)

The equation $K\overline{K} = I$ yields

$$ |\alpha|^2 = |\beta|^2 = 1, \quad \alpha \beta + \beta \overline{\alpha} = 0 $$

The last relation means that $\beta = i\alpha \alpha$ for some real scalar $\alpha$. Setting $\phi = \arg \alpha$, we see that matrix (18) can be written in the form

$$ K = e^{i\phi} \begin{pmatrix} \cosh t & isinh t \\ isinh t & \cosh t \end{pmatrix}, \quad \phi \in [0, 2\pi), \quad t \in R. $$

In view of the equality $|\det K| = 1$, the moduli of the s-eigenvalues of $K$ are reciprocal numbers. Excluding the case $t=0$, these moduli differ from unity because the sum of the s-eigenvalues has the modulus 2cos$h>2$.

### 4.1.2. Reduction to a direct sum

A complete description of the s-normal solutions to Eq.(3) will be based on the following lemma:

**Lemma 1.** Let $K$ be an arbitrary solution to Eq.(3), where $n \geq 3$. Then, there exists a $K$-invariant subspace of dimension one or two with a basis formed of vectors with real components.

**Proof.** Let $\lambda$ be an arbitrary s-eigenvalue of $K$, and let $z$ be the associated s-eigenvector. If $z$ is a multiple of a real vector, then span $\{z\}$ is the desired one-dimensional subspace. Otherwise, $z$ and $\overline{z}$ are linearly independent. Moreover, the relation $Kz = \lambda z$ and Eq.(3) imply that $K\overline{z} = \overline{\lambda} z$ and Eq.(3) imply that $K\overline{z} = \overline{\lambda} z$. Thus, $\overline{z}$ is also an s-eigenvector of $K$, which corresponds to the s-eigenvalue $1/\overline{\lambda}$. The two-dimensional subspace spanned by $z$ and $\overline{z}$ is invariant with respect to $K$ and has a real basis formed of the vectors $u = z + \overline{z}$ and $v = i(z - \overline{z})$.

**Remark 1.** We stress that the assertion of the lemma is valid for all the solutions of Eq.(3). The proof given above by no means uses s-normality, which is a property of interest to us.

**Corollary 1.** Let $K$ be an s-normal solution to Eq.(3), where $n \geq 3$. Then, $K$ can be brought by a real $s$-orthogonal similarity transformation to the form $K_{11} \oplus K_{22}$, where $K_{11}$ is a block of order one or two and $K_{22}$ again satisfies Eq.(3) for an appropriate $n$.

A repeated application of this corollary yields the desired description of s-normal solutions to Eq.(3).

**Theorem 4.** Let $K$ be an s-normal solution to Eq.(3), where $n \geq 3$. Then, $K$ can be brought by a $s$-orthogonal similarity transformation to the form

$$ K_{11} \oplus \ldots \oplus K_{nn} $$

(19)

Where all the blocks are 1-by-1 or 2-by-2 and satisfy Eq.(3) for appropriate $n$.

Each block in direct sum (19) has one of the forms described in the preceding subsection.

Every real $s$-orthogonal similarity transformation preserves both the s-normality of a matrix and the form of Eq.(3) (because such a transformation is a congruence). Consequently, the converse assertion to Theorem 4 is also true.

**Theorem 5.** Let $\bar{K}$ be an arbitrary direct sum of the type described in Theorem 4. Then, every matrix of the form

$$ \bar{K} = QKQ^o $$

(20)

where

$$ Q \in M_1(R), \quad QQ^o = I $$

is a s-normal solution to Eq.(3).

### 4.2. Equation (4)

Again, almost the entire analysis given in the preceding subsection remains valid for Eq.(4).

**4.2.1. $n=1$ and $n=2$.** For $n=1$, Eq.(4) takes the form

$$ kk = |k|^2 $$

And, therefore, has no solutions.

Assume that $n=2$. Let us find the solutions to the system of matrix equations

$$ K\overline{K} = -I, \quad KK^o = K^o K $$

As before, the second equation in this system can be rewritten in simpler form (15). Consequently, the entire analysis related to this equation remains valid. Adopting representation (16) for the desired matrix, we arrive at the relation $\beta = \pm \gamma$. However, the case $\beta = \gamma$ is now impossible because, otherwise, we would have $K = K^S$ and $K\overline{K} = KK^o = -I$. Obviously, there is no matrix $K$ that can satisfy the last equality.
Thus, assume that $\beta = -\gamma \neq 0$. Using representation (18) of $K$, we derive from the equation $K\overline{K} = -I$ the scalar relations

$|\nu|^2 - |\rho|^2 = -1$

and

$\text{Re}(\alpha \overline{\beta}) = 0$

Again, we have $\beta = i\alpha a$ for some real scalar $a$. Setting $\phi = \arg\alpha$, we see that matrix (18) can be written in the form

$K = -e^{\phi} \begin{pmatrix} -\sinh t & -i\cosh t \\ i\cosh t & -\sinh t \end{pmatrix}$,  $\phi \in [0, 2\pi)$,  $t \in \mathbb{R}$

In view of the equality $|\det K| = 1$, the moduli of the $s$-eigenvalues of $K$ are reciprocal numbers.

**4.2.2. Reduction to a direct sum.** A complete description of the $s$-normal solutions to Eq.(4) will be based on the following lemma:

**Lemma 2.** Let $K$ be an arbitrary solution to Eq.(4), where $n > 2$. Then, there exists an $K$-invariant subspace of dimension two with a basis formed of vectors with real components.

**Proof.** Let $\lambda$ be an arbitrary $s$-eigenvalue of $K$. The associated $s$-eigenvector $z$ cannot be a multiple of a real vector because, otherwise, $\lambda$ would satisfy the relation $\lambda^2 = -1$. Therefore, $z$ and $\overline{z}$ are linearly independent vectors. Moreover, the relation $Kz = \lambda z$ and Eq.(4) imply that

$K\overline{z} = \overline{\lambda} z = \overline{\lambda} \overline{z} \oplus K\overline{z} = -\frac{1}{\lambda} z$.

Thus, $\overline{z}$ is also an $s$-eigenvector of $K$, which corresponds to the $s$-eigenvalue $1/\lambda$. The two-dimensional subspace spanned by $z$ and $\overline{z}$ is invariant to $K$ and has a real basis formed of the vectors.

$u = z + \overline{z}$,  $v = i(z - \overline{z})$.

**Remark 2.** We stress that the assertion of the lemma is valid for all the solutions to Eq.(4). The proof given above 5.

by no means uses $s$-normality, which is a property of interest to us.

**Corollary 2.** Let $K$ be an $s$-normal solution to Eq.(4), where $n > 2$. Then, $K$ can be brought by a real $s$-orthogonal $s$-similarity transformation to the form $K_{11} \oplus K_{22}$, where $K_{11}$ is a block of order two and both $K_{11}$ and $K_{22}$ again satisfy Eq.(4) for appropriate $n$.

A repeated application of this corollary yields the desired description of $s$-normal solutions to Eq.(4).

**Theorem 6.** Let $K$ be a $s$-normal solution to Eq.(4), where $n > 2$. Then, $K$ can be brought by a real $s$-orthogonal $s$-similarity transformation to the form

$K_1 \oplus \cdots \oplus K_n$.

(21)

Where all the blocks are 2-by-2 and satisfy Eq.(4) for $n=2$.

Each block in direct sum (21) has the form described in the preceding subsection.

Every real $s$-orthogonal $s$-similarity transformation preserves both the $s$-normality of a matrix and the form of Eq.(4) (because such a transformation is a congruence). Consequently, the converse assertion to Theorem 6 is also true.

**Theorem 7.** Let $\overline{K}$ be an arbitrary direct sum of the type described in Theorem 6. Then, every matrix of form (20), where

$Q \in M_n(R)$,  $QQ^s = I_n$, is an $s$-normal solution to Eq.(4).

**REFERENCES**


