CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS CONDITIONED ON A PAIR OF NON-ADJACENT DUAL GENERALIZED ORDER STATISTICS USING MEIJER’S G-FUNCTION

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INTRODUCTION

The concept of generalized order statistics (gos) has been introduced as a unified approach to a variety of models of ordered random variables with different interpretation (Kamps, 1995), such as ordinary order statistics, sequential order statistics, progressive type II censoring, record values and Pfeifer’s records. Generalized order statistics serve as a common approach to a structural similarities and analogies. Several of these models can be effectively applied, e.g., in reliability theory. Although gos contains many useful models of ordered random variables, the random variables that are decreasingly ordered cannot be integrated into this framework. Burkschat et al. (2003) introduced the concept of dual generalized order statistics (dgos) as a systematic approach to some models of descending ordered random variables. Dual gos represent a unification of models of decreasingly ordered random variables, e.g., reversed ordered order statistics, lower records, lower k-records, and lower Pfeifer records.

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Further, let $n \in \mathbb{N}$, $n \geq 2$, $k \geq 1$, $\bar{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{N}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 1$, for all $r \in \{1, 2, \ldots, n-1\}$. Then $X^*(r, n, \bar{m}, k)$, $r = 1, 2, \ldots, n$ are called dgos if their joint pdf is given by

$$ f \left( \prod_{j=1}^{n-1} \gamma_j \prod_{i=1}^{n-1} [F(x_i)]^{\gamma_i} f(x_i) [F(x_n)]^{\gamma_n-1} f(x_n), \right) \quad (1.1) $$

for $F^{-1}(1) > x_1 \geq x_2 \geq \ldots \geq x_n > F^{-1}(0)$.

The various developments on dual generalized order statistics and related topic have been studied by Alshanallah (2004), Mbah and Alshanallah (2007), Khan et al. (2009), Khan et al. (2010 a,b), Faizan and Khan (2011), Tavangar (2011) amongst others. Khan et al. (2009) have characterized continuous distributions through conditional expectation of dgos, conditioned on a pair of non-adjacent dgos.

Recently Khan and Khan (2012) have characterized continuous distribution functions conditioned on non-adjacent dgos using Meijer’s G-function. In this paper we have extended the result of Khan and Khan (2012) when the regression is based on two non-adjacent dgos. Also the result is deduced for known results on dgos.

Let $P_F$ stand for the probability measure on $\mathbb{R}$ determined by $F(x)$, then the pdf of $X^*(r, n, \bar{m}, k)$ with respect to a measure $P_F$ is given

$$ f_r(x) = C_{r-1} G_r(F(x)|\gamma_1, \ldots, \gamma_r) I_{(\alpha, \beta)}(x). \quad (1.2) $$

where $I_A$ denotes the indicator function over the support of $A$ and $G_r(x) = G_r,0(x|\gamma_1, \ldots, \gamma_r)$

$$ = G_{r,r}(0,1,\ldots,\gamma_r | \gamma_1, \ldots, \gamma_{r-1}) $$

is the particular Meijer’s G-function defined by
The joint \( P_{T_0} \otimes P_{T_1} \) density of \( X^* (r, n, \tilde{m}, k) \) and 
\[ X^* (s, n, \tilde{m}, k), \quad 1 \leq r < s \leq n, \] 
is given by 
\[ f_{r,s} (x, y) = c_{r,s} G_{r,s} \left( \frac{F(y)}{F(x)} \right) (r + 1, \ldots, s) \]
and \( G_{r,s} \) is defined by 
\[ G_{r,s} (x) = \sum_{j=1}^{3} \frac{1}{2\pi i} \int_{c} \frac{s^j}{(x^j - 1)^{r+j-1}} dz, \quad j = 1, \ldots, s, \] 
where \( c \) is an appropriate chosen contour of integration. See Mathai (1993, Chapter 3) for the definition of \( G \)-function and its numerous properties and applications.

The joint \( P_{T_i} \otimes P_{T_j} \) density of \( X^* (r, n, \tilde{m}, k) \) and 
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where \( c \) is an appropriate chosen contour of integration. See Mathai (1993, Chapter 3) for the definition of \( G \)-function and its numerous properties and applications.

2. CHARACTERIZATION OF DISTRIBUTIONS

Theorem 2.1: Let \( X^* (i, n, \tilde{m}, k), \quad i = 1, \ldots, n \) be the \( i \)-th 
dg function of a continuous population with the \( df \) \( f(x) \) and the \( pdf \) \( f(x) \) over the support \((a, b)\), and \( h(t) \) be a monotonic and differentiable function of \( t \). If for two consecutive values \( r \) and \( r+1 \), 
\[ 1 < r + 1 < J < s \leq n, \]
and \( g(\cdot) \) is a finite and differentiable function of \( x \), and 
\[ D_t (x) = \frac{g(x) - g(x + J)}{J}, \] 
then
\[ g_{r,s} (x, y) = \frac{1}{r} \sum_{j=1}^{r} \frac{1}{2\pi i} \int_{c} \frac{s^j}{(x^j - 1)^{r+j-1}} dz, \quad j = 1, \ldots, s, \] 
where \( c \) is an appropriate chosen contour of integration. See Mathai (1993, Chapter 3) for the definition of \( G \)-function and its numerous properties and applications.

Corollary 2.1: Using residue theorem, it can be proved that 
\[ G_{r,s \rightarrow} \left( \frac{F(x)}{F(x)} \right) (r + 1, \ldots, s) = \sum_{j=1}^{s} \phi_j (x) \left( \frac{F(x)}{F(x)} \right)^{r+j-1}, \]
where \( a_i^{(r)}(s) = \sum_{j=r+1}^n \frac{1}{(y_j - \gamma_i)} \), \( \gamma_j \neq \gamma_i \), \( r+1 \leq i \leq n \)

Therefore (2.1) reduces to
\[
\frac{[F(x)]^{r-1} B_s^{(r)}(x,y)}{B_s^{(m)}(x,y)} = \exp \left[ \int_{x}^{\gamma_i} \beta D_1(t,y)dt \right],
\]
(2.5)
where
\[
B_s^{(r)}(x,y) = \sum_{i=r+1}^n a_i^{(r)}(s) \frac{F(x)^{\gamma_i}}{F(x)}
\]
as obtained by Khan et al. (2009).

Also since for \( m_1 = \ldots = m_{n-1} = m \neq -1 \),
\[
a_i^{(r)}(s) = \frac{1}{(\gamma_j - y_j)} = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \begin{pmatrix} \frac{s-r-1}{s-i} \end{pmatrix},
\]

thus (2.1) for \( m_1 = \ldots = m_{n-1} = m \neq -1 \), reduces to
\[
1 - [F(x)]^{m+1} = 1 - \exp \left[ -\frac{1}{(s-r-1)} \int_{x}^{\gamma_i} \beta D_1(t,y)dt \right], \quad m \neq -1.
\]
(2.6)
and
\[
\log F(x) = 1 - \exp \left[ -\frac{1}{(s-r-1)} \int_{x}^{\gamma_i} \beta D_1(t,y)dt \right], \quad m = -1
\]
(2.7)
as obtained by Khan et al. (2009).

**Remark 2.1:** At \( \gamma_j = 0 \) i.e. \( s = k + n + M \), by convention \( X^*(s, n, \tilde{m}, k) = y = \alpha \) and hence \( F(\alpha) = 0 \). Therefore, \( g_j|_{\gamma_j}(x) = E[h(X^{(j,n,\tilde{m},k)}) X^*(r, n, \tilde{m}, k) = x] \),

and
\[
F(x) = \exp \left[ \frac{1}{\gamma_{r+1}} \int_{x}^{\gamma_{r+1}} \left[ g_j|_{r+1}(t) - g_j|_{r}(t) \right] dt \right]
\]
(2.8)
as given by Khan et al. (2010a), Khan and Khan (2012). The result for lower record is given by
\[
F(x) = \exp \left[ -\frac{\beta}{x} \int_{x}^{\gamma_{r+1}} \left[ g_j|_{r+1}(t) - g_j|_{r}(t) \right] dt \right]
\]
Theorem 2.2: Let \( X^*(i, n, \tilde{m}, k) \), \( i = 1, \ldots, n \) be the \( i^{\text{th}} \) dgos from a continuous population with the df \( F(x) \) and the pdf \( f(x) \) over the support \((\alpha, \beta)\), and \( h(t) \) is a monotonic and differentiable function of \( t \). If for two consecutive values \( s-1 \) and \( s, 1 \leq r < j < s-1 < n \),
\[
g_j|_{\gamma_j}(x,y) = E[h(X^{(j,n,\tilde{m},k)}) X^*(r, n, \tilde{m}, k) = x, X^*(l, n, \tilde{m}, k) = y], l = s-1, s exists, then
\]
\[
(\gamma_j - 1) \frac{f(y)}{F(y)} - \frac{\partial}{\partial y} G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s = \frac{\partial}{\partial y} g_j|_{r,s}(x,y) = \frac{[g_j|_{r,s}(x,y) - g_j|_{r,s-1}(x,y)]}{[g_j|_{r,s}(x,y) - g_j|_{r,s-1}(x,y)]},
\]
(2.9)

and
\[
G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s = a_s^{(r)}(s) \exp \left[ \int_{x}^{\gamma_i} \beta D_2(x,t)dt \right], \quad \forall \gamma_i > \gamma_s, i = r+1, \ldots, s-1
\]
(2.10)
and for \( \gamma_{r+1} = \ldots = \gamma_s \),
\[
1 + \log F(y) = 1 - \exp \left[ -\frac{1}{(s-r-1)} \int_{x}^{\gamma_s} \beta D_2(x,t)dt \right]
\]
(2.11)
where \( p \in (\alpha, \beta) : -\log F(p) = 1 \),
(2.12)
and
\[
D_2(x,y) = \frac{\partial}{\partial y} g_j|_{r,s}(x,y)
\]
(2.13)

**Proof:** Differentiating both the sides of (2.3) \( w.r.t. \), \( y \) and proceeding as in the Theorem 2.1, we get
\[
f(y) G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s = \frac{\partial}{\partial y} g_j|_{r,s}(x,y)
\]
(2.14)
It can be seen that for \( \gamma_j > \gamma_s, i = r+1, \ldots, s-1 \),
\[
G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s \quad \text{and therefore}
\]
\[
G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s
\]
(2.15)
and
\[
a_s^{(r)}(s) \exp \left[ \int_{x}^{\gamma_i} \beta D_2(x,t)dt \right]
\]
(2.16)
also, for \( \gamma_{r+1} = \ldots = \gamma_s \), (Cramer, 2002, p. 35)
\[
G_{s-r} \left( \frac{F(y)}{F(x)} \right) \gamma_{r+1} \ldots \gamma_s
\]
(2.17)
Thus in the case of lower record statistic,
\[
1 + \log F(y) = 1 - \exp\left(-\frac{1}{(s-r-1)\alpha}\int_{p}^{y}D_2(x,t)\,dt\right),
\]  
(2.15)
where \( p \) is as defined in (2.12).

**Corollary 2.2:** It may be noted that if \( \gamma_i \neq \gamma_j \) but
\[ m_1 = \ldots = m_{n-1} = m > -1, \]
\[ g_{\gamma_i}(\frac{\alpha(x)}{\alpha(x+1)})^{\gamma_i-\gamma} = \sum_{j=1}^{r} g_{\gamma_j}(\frac{\alpha(x)}{\alpha(x+1)})^{\gamma_j-\gamma} \]
Thus (2.8) reduces to
\[
\frac{[F(y)]^{m+1}}{[F(x)]^{m+1}} = 1 - \exp\left(-\frac{1}{(s-r-1)\alpha}\int_{\alpha}^{y}D_2(x,t)\,dt\right),
\]  
(2.16)
as obtained by Khan et al. (2010a).

**Remark 2.2:** With the convention \( X^*(0, n, \bar{m}, k) = x = \beta, \) \( r = 0, \) Theorem 2.2 reduces to
\[
g_{\gamma_i}(F(y)|\gamma_i, \gamma_j, \ldots, \gamma_r, 1) + \sum_{i=1}^{r} g_{\gamma_j}(F(y)|\gamma_i, \gamma_j, \ldots, \gamma_r, 1) = 0 \quad \text{if} \quad \gamma_i > \gamma_j, \]
(2.17)
and for \( m_1 = \ldots = m_{n-1} = m > -1, \)
\[ F(y)^{m+1} = 1 - \exp\left(-\frac{1}{(s-r-1)\alpha}\int_{\alpha}^{y}D_2(x,t)\,dt\right), \]  
(2.18)
whereas for \( \gamma_{r+1} = \ldots = \gamma_k, \)
\[ -\log F(y) = \exp\left(-\frac{1}{(s-r-1)\alpha}\int_{\alpha}^{y}D_2(x,t)\,dt\right), \]  
(2.19)
where \( p \) is as defined in (2.12), and
\[
D(y) = \frac{g_{\gamma_i}(y)}{g_{\gamma_j}(y) - g_{\gamma_i}(y)} = D_2(\alpha, y),
\]  
as obtained by Khan et al. (2010a), Khan and Khan (2012).

**Corollary 2.3:** Under the assumptions given in Corollary 2.1 and Corollary 2.2,
\[
F(x) = \left[\frac{e^{l_1}}{e^{l_1} + e^{l_2} - 1}\right]^{m+1}, \quad m > -1 \]  
(2.20)
and
\[
F(y) = \left[\frac{e^{l_1} - 1}{e^{l_1} + e^{l_2} - 1}\right]^{m+1}, \quad m > -1 \]  
(2.21)
where
\[ I_1 = \int_{x}^{y} A_1(t,y)\,dt, \quad I_2 = \int_{\alpha}^{y} A_2(x,t)\,dt \]
and
\[ A_1(x,y) = \frac{D_1(x,y)}{(s-r-1)}, \quad A_2(x,y) = \frac{D_2(x,y)}{(s-r-1)}. \]
Similar result for lower records can be obtained.

**REFERENCES**