ON A CONTINUED FRACTION IDENTITY FROM RAMANUJAN’S NOTEBOOK

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ABSTRACT

In this paper, we study a continued fraction of Ramanujan A (q). We establish an integral representation of A (q) and prove its modular identities. We also compute explicit evaluation of this continued fraction. 2000 Mathematics Subject Classification: 11A55.

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Introduction

Ramanujan recorded about 200 results on continued fractions in his notebooks [11] and lost notebook [12] without proof. The only result on continued fraction that he published [9], [10, pp.214-215], is related to the now celebrated Roger Ramanujan continued fraction defined by

\[ R(q) = \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}, \quad |q| < 1, \]

\[ S(q) = R(-q), \]

which was first introduced by L.J. Roger [13] and independently rediscovered by Ramanujan. In his first two letters to G.H.Hardy [10, pp. xxvii-xxviii] [8, pp.21-30,53-62], Ramanujan communicated several results concerning R (q). In particularly, he asserted that

\[ R(e^{-2\pi q}) = \frac{\sqrt{5+\sqrt{5}}}{2} \frac{\sqrt{5}+1}{2}, \]

\[ S(e^{-\pi q}) = \frac{\sqrt{5-\sqrt{5}}}{2} \frac{\sqrt{5}-1}{2}. \]

which were first proved by G.N. Watson [15]. In his lost notebook [12, pp.26], Ramanujan claims that

\[ R(q) = \frac{\sqrt{5}-1}{2} \exp\left(\frac{-1/5}{\int_q^1 (1-t^5)(1-t^{10})^{-t}dt}\right). \] (1.1)

(391)
\[
\frac{\sqrt{5} - 1}{2} - \frac{\sqrt{5}}{1 + 2\sqrt{5}} \exp \left( \frac{1}{5} \int_{q}^{1} \frac{(1-t)^5(1-t^2)^5 \ldots \text{dt}}{(1-t^5)(1-t^{10})} \right), \quad (1.2)
\]

where 0 < q < 1. The equality (1.1) was first proved by G.E. Andrews [3] and equality (1.2) was proved by S.H. Son [14]. On page 365 of his lost notebook [12], Ramanujan recorded five modular equations relating \( R(q) \) with \( R(-q) \), \( R(q^2) \), \( R(q^3) \), \( R(q^4) \) and \( R(q^5) \).

Motivated by these works in this paper we study the Ramanujan continued fraction

\[
R(q) = \frac{1}{1 - q^2 + \frac{q^2(1+q^2)^2}{1 - q^6} + \frac{q^4(1+q^4)^2}{1 - q^{10}} + \frac{q^6(1+q^6)^2}{1 - q^{14}} + \ldots}, \quad |q| < 1, \quad (1.3)
\]

A similar continued fraction is been previously studied by C. Adiga and N. Anitha [1].

In Section 2 we obtain an interesting q-identity related to \( A(q) \) using Ramanujan’s \( _1\psi_1 \) summation formula [5, Ch.16, Entry 17]

\[
\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/az)_\infty (b)_\infty (q/a)_\infty}, \quad \frac{|b|}{|a|} < |z| < 1. \quad (1.4)
\]

In Section 3 we obtain a product representation for \( A(q) \). In Section 4 we deduce several identities satisfied by \( A(q) \) with theta function \( \varphi(q) \) and \( \psi(q) \). In Section 5 we obtain an integral representation of \( A(q) \). Section 6 contains relationship of \( A(q) \) with ordinary hyper geometric series. We discuss modular equations of degree \( n \) with some illustration in Section 7. Finally, in Section 8 we present formula for explicit evaluation of \( A(q) \).

We conclude this introduction with few customary definition we make use in the sequel. For a and \( q \) complex number with \(|q| < 1\)

\[
(a)_\infty := (a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n),
\]

\[
(a)_n := (a; q)_n = \prod_{k=0}^{n-1} (1 - a q^k) = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n : \text{any integer},
\]

\[
f(a, b) = \sum_{n=0}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}
\]

\[
= (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad |ab| < 1. \quad (1.6)
\]

Identity (1.6) is the Jacobi’s triple product identity in Ramanujan’s notation [5, Ch. 16, Entry 19]. It follows from (1.5) and (1.6) that [5, Ch.16, Entry 22],

\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^n = \frac{(-q; q)_{\infty}}{(q; -q)_{\infty}}, \quad (1.7)
\]

and
\[ \psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty}. \]  

(1.8)

2. q-Identity Related to A (q)

Theorem 2.1

ii. \[ A(q) = \frac{(q^4;q^4)_\infty^2}{(q^8;q^8)_\infty} \sum_{n=0}^{\infty} \frac{q^{4n}}{1+q^{4n+1}}. \]  

(2.1)

**Proof:** Changing \( q \) to \( q^2 \), then setting \( a = -q^4 \), \( b = -q^{12} \) and \( z = q^4 \) in Ramanujan's \( \psi \) summation formula (1.4), we complete the proof of Theorem 2.1.

3. Product Representation for A (q)

Theorem 3.1. Let \( A(q) \) be defined by (1.3). Then

\[ A(q) = \frac{\psi^2(q^8)}{\psi^2(q^4)}. \]  

(3.1)

**Proof:** From [5, Ch. 16, Entry 11], for \( |q| < 1 \)

\[ \frac{(-a)\omega(b) = -(a)\omega(-b) \omega a-b}{(-a)\omega(b) \omega + (a)\omega(-b) \omega} = \frac{(a-bq)(aq-b)}{1-q + \frac{1-q^3}{1-q^5} + \cdots}. \]  

(3.2)

Rationalizing left hand side of (3.2) and then changing \( q \) to \( q^2 \), \( a \) to \( q \) and \( b \) to \( -q \) in the resulting identity, we obtain

\[ \frac{\{(q^2;q^2)_\infty^2 - (q^2;q^2)_\infty^2\}^2}{(-q^2;q^2)_\infty^2} = 2q \frac{q^2(1+q^2)^2}{1-q^2} \frac{q^4(1+q^4)^2}{1-q^6} + \cdots. \]  

(3.3)

Multiplying numerator and denominator of left hand side of (3.3) by \((q^2;q^2)_\infty^2\) and using (1.6), we obtain

\[ \frac{(f(q,q) - f(-q,-q))^2}{f^2(q,q) - f^2(-q,-q)} = 2q \frac{q^2(1+q^2)^2}{1-q^2} \frac{q^4(1+q^4)^2}{1-q^6} + \cdots. \]  

(3.4)

Employing [5, Ch. 16, Entry 30 (iii) and (vi)] in (3.4) we obtain

\[ \frac{qf^2(1,q^8)}{f(1,q^4)\psi(q^4)} = 2q \frac{q^2(1+q^2)^2}{1-q^2} \frac{q^4(1+q^4)^2}{1-q^6} + \cdots. \]  

(3.5)

Finally applying [5, Ch. 16, Entry 18 (ii)] and (1.8), we complete the proof of (3.1).

4. Some Identities Involving A (q)

We obtain several relations of \( A(q) \) in terms of theta functions \( \varphi(q) \) and \( \psi(q) \).

Theorem 4.1

\[ A(q) = \frac{\psi^2(q^4)}{\psi^2(q^4)}. \]  

(4.1)

\[ A(q) = \frac{\psi(q^8)}{\varphi(q^8)}. \]  

(4.2)

\[ A(q) = \frac{\psi(-q^2)}{\varphi(-q^2)\varphi(q^4)}. \]  

(4.3)
\[ A(q) = \psi^2(q^4) \varphi(-q^8) \varphi(q^4)^3. \]  
(4.12)

Proof: From (3.1) we have
\[ uv = A(q) A_{\frac{1}{2}}(q) = \left( \frac{\psi(q^8)}{\psi(q^4)} \right)^3, \]
which completes the proof of (4.12).

Also from (3.1), we have
which completes the proof of (4.13).

5. Integral Representation of A (q)

**Theorem 5.1** For 0<|q|<1,

\[
A(q) = \exp \int \frac{\psi(q^8)}{\psi(q^4)} dq,
\]

(5.1)

where \(\psi(q)\) is defined in (1.7).

**Proof:** Taking log on both sides of (3.1), we have

\[
\log A(q) = 2 \log \psi(q^8) - 2 \log \psi(q^4).
\]

(5.2)

Employing [5, Ch. 16, Entry 23(ii)] on right hand side of (5.2), we obtain

\[
\log A(q) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{4n}}{n(1+q^{4n})}.
\]

(5.3)

Differentiating (5.3) and simplifying, we have

\[
\frac{d}{dq} \log A(q) = \frac{8}{q} \sum_{n=1}^{\infty} \frac{(-1)^n q^{4n}}{(1+q^{4n})^2}.
\]

(5.4)

Using Jacobi’s identity [5, Ch. 16, Identity 33.5, pp. 54] and integrating both sides and finally exponentiating both sides of identity (5.4), we complete the proof of Theorem 4.1.

6. Relation Between A (q) and Hyper geometric Function

In this section we deduce relations between A (q) and hyper geometric function

\[
_2F_1(a, b; c; x),
\]

where

\[
_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k,
\]

|x|<1

(6.1)

**Theorem 6.1**

\[
x = k^2 = \frac{\psi(4q)}{\psi(q)}
\]

\[
q = \exp(-\pi 2F_1(1/2, 1/2; 1, 1-k^2)/2F_1(1/2, 1/2; 1, k^2),
\]

and

\[
z = _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)
\]

then

(i) \(A(q) = \frac{1}{q}\).
(ii) \( A(q) + \frac{1}{2q} \frac{1}{q^x} \left( 1 + \sqrt{1-x} - 4 \sqrt[4]{(1-x)^3} \right). \)

**Proof of (i):** From [5, Ch. 17, Entry 11 (i), pp. 123], we have
\[
\psi(q) = \sqrt{\frac{1}{2} z(\frac{x}{q})^{1/8}}. \tag{6.2}
\]
Then employing (6.2) in (3.1) we obtain (i).

**Proof of (ii):** From [5, Ch. 17, Entry 11 (iv), (v), pp. 123] and (3.1), we have
\[
A(q) = \frac{1}{2q} \left( \frac{1-(\sqrt{1-x})}{1-\sqrt{1-x}} \right)^2. \tag{6.3}
\]
Rationalizing (6.3) and simple manipulation completes the proof of (ii).

### 7. Modular Equations of Degree \( n \)

In this section we obtain new modular equations of \( B(q) \), where \( B(q) = 2q A(q) \).

We say modulus \( \beta \) has degree \( n \) over the modulus \( \alpha \) when
\[
n \frac{\text{F}_2(1/2,1/2;;1-\alpha)/\text{F}_2(1/2,1/2;1:\alpha)}{\text{F}_2(1/2,1/2;;1-\beta)/\text{F}_2(1/2,1/2;1:\beta)} = \frac{\text{F}_2(1/2,1/2;;1-\alpha)}{\text{F}_2(1/2,1/2;1:\alpha)} = \frac{1}{\text{F}_2(1/2,1/2;;1-\beta)/\text{F}_2(1/2,1/2;1:\beta)}. \tag{7.1}
\]
where \( \text{F}_2(a, b; c; x) \) is defined as in (6.1). A modular equation of degree \( n \) is an equation relating \( \alpha \) and \( \beta \) induced by (7.1).

**Theorem 7.1:**

If \( q = \exp(-\pi \frac{\text{F}_2(1/2,1/2;1-\alpha)/\text{F}_2(1/2,1/2;1:\alpha)}{\text{F}_2(1/2,1/2;;1-\beta)/\text{F}_2(1/2,1/2;1:\beta)}) \),
then
\[
\alpha = 1 - \frac{1-B(q)}{1+B(q)}. \tag{7.3}
\]

**Proof:** On employing [5, Ch. 16, Entry 25 (ii)] and [5, Ch. 16, Entry 25(v)] in (3.1), we have
\[
A(q) = \frac{1}{2q} \left( \frac{1-\frac{\psi(-q)}{\psi(q)}}{1+\frac{\psi(-q)}{\psi(q)}} \right).
\]
Thus
\[
B(q) = \left( \frac{1-\frac{\psi(-q)}{\psi(q)}}{1+\frac{\psi(-q)}{\psi(q)}} \right). \tag{7.4}
\]
Also from [5, Ch. 17, Entry 5, pp. 100] and (7.2) it is implied that
\[
\alpha = 1 - \frac{\psi^n(-q)}{\psi^n(q)}. \tag{7.5}
\]
Using (7.5) in (7.4), we complete the proof of (7.3).

Let \( q \) and \( \alpha \) is related by (7.2). If \( \beta \) has degree \( n \) over \( \alpha \) then from Theorem 5.1, we obtain
\[
\beta = 1 - \left( \frac{1-B(q^n)}{1+B(q^n)} \right)^4 .
\]  
(7.6)

**Corollary 7.2** Let \( l = B(q) \), \( m = B(q^3) \), \( n = B(q^4) \), then

(i) \( l^4 - 4l^3m^3 + 6l^2m^2 - 4lm + m^4 = 0 \).

(ii) \( l^4 + l^4n^4 + 4l^4n^3 + 6l^4n^2 + 4l^4n - 8n^3 - 8n = 0 \).

**Proof of (i):** From [5, Ch. 19, Entry 5 (ii) pp. 230], we have

\[
(\alpha \beta)^{\frac{1}{4}} + [(1 - \alpha)(1 - \beta)]^{\frac{1}{4}} = 1 \]  
(7.7)

On using (7.6) with \( n=3 \) and (7.3) in (7.7) and simplifying we complete the proof of Corollary 7.2. (i).

**Proof of (ii):** When \( \beta \) has degree 4 over \( \alpha \) then we have from [5, Ch. 18, Eq. (24.22) pp.215]

\[
\sqrt{\beta} = \left( \frac{1-(\alpha)^{\frac{1}{4}}}{1+(\alpha)^{\frac{1}{4}}} \right)^2 .
\]  
(7.8)

On using (7.6) with \( n=4 \) and (7.3) in (7.8), we obtain

\[
\sqrt{1 - \left( \frac{1-n}{1+n} \right)^4} = \left( \frac{1-\left(\frac{1}{\alpha_{n}}\right)^{\frac{1}{4}}}{1+\left(\frac{1}{\alpha_{n}}\right)^{\frac{1}{4}}} \right)^2 .
\]

Squaring both sides of the above identity and simplifying we complete the proof of Corollary 7.2. (ii).

8. **Explicit Formula For The evaluation of A (q)**

Let \( q_n = e^{-\pi \sqrt{n}} \) and \( \alpha_n \) denote the corresponding values of \( \alpha \) in (7.2). From Theorem 7.1., we have

\[
B(e^{-\pi \sqrt{n}}) = \frac{1-(1-\alpha_n)^{\frac{1}{4}}}{1+(1-\alpha_n)^{\frac{1}{4}}} .
\]

Hence

\[
A(e^{-\pi \sqrt{n}}) = \frac{1}{2} e^{\pi \sqrt{n}} \frac{1-(1-\alpha_n)^{\frac{1}{4}}}{1+(1-\alpha_n)^{\frac{1}{4}}} .
\]  
(8.1)

From [5, Ch. 17, pp. 97], we have \( \alpha_1 = \frac{1}{2}, \alpha_2 = \left( \sqrt{2} - 1 \right)^{2}, \alpha_4 = \left( \sqrt{2} - 1 \right)^{4} \). Thus from (8.1), we have

\[
A(e^{-\pi}) = \frac{1}{2} e^{\pi} \frac{1-\left(\frac{1}{2}\right)^{\frac{1}{4}}}{\frac{1}{2} + \left(\frac{1}{2}\right)^{\frac{1}{4}}} .
\]

\[
A(e^{-\pi \sqrt{2}}) = \frac{1}{2} e^{\pi \sqrt{2}} \left( \frac{1-\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{4}}}{\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{4}}} \right),
\]

\[
A(e^{-2\pi}) = \frac{1}{2} e^{2\pi} \left( \frac{1-\left(-16+12\sqrt{2}\right)^{\frac{1}{4}}}{\frac{1}{16+12\sqrt{2}} + \left(\frac{1}{16+12\sqrt{2}}\right)^{\frac{1}{4}}} \right).
\]

The Ramanujan-Weber class invariant is defined by

\[
G_n = 2 \frac{1}{4} q_n^{-\frac{1}{24}} (-q_n; q_n^2)_\infty ,
\]
\[ g_n = 2^{-\frac{1}{4}} q_n^{-\frac{1}{24}} (q_n; q_n^2)_\infty, \]

where \( q_n = e^{-\pi \sqrt{n}} \). Chan and Huang [7] have derived few explicit formula for evaluation of

\( S \left( e^{-\pi \sqrt{n/2}} \right) \) in terms of Ramanujan Weber class. Similar works are also carried out by Adiga et. al. [2]. Analogous to these works we obtain explicit formula for the evaluation of \( A(e^{-\pi \sqrt{n}}) \).

**Theorem 8.1**

For Ramanujan Weber class invariant as defined in (8.2) and let \( p = G_n^{12} \) and \( p_1 = g_n^{12} \), then

\[
A \left( e^{-\pi \sqrt{n}} \right) = \frac{1}{2} e^{\pi \sqrt{n}} \left[ \frac{(2p(p + \sqrt{p^2 + 1} - 1))^{1/4} - (2p(p + \sqrt{p^2 + 1} - 1))^{1/4}}{(2p(p + \sqrt{p^2 + 1} - 1))^{1/4} + (2p(p + \sqrt{p^2 + 1} - 1))^{1/4}} \right], \tag{8.3}
\]

\[
A \left( e^{-\pi \sqrt{n}} \right) = \frac{1}{2} e^{-\pi \sqrt{n}} \left[ \frac{1 - (2p_1(p_1 - [p_1^2 + 1]))^{1/4}}{1 + (2p_1(p_1 - [p_1^2 + 1]))^{1/4}} \right]. \tag{8.4}
\]

**Proof:** From [7, Eq. 4.7, pp. 85], we have

\[
G_n = \{4\alpha_n (1 - \alpha_n)\}^{-1/24}.
\]

Hence,

\[
\alpha_n = \frac{1}{\sqrt{p(p+1) + \sqrt{p(p-1)}}}. \tag{8.5}
\]

Using (8.5) in (8.1) we obtain (8.3).

Also from [7, Eq. 4.9, pp. 85], we have

\[
2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}.
\]

Hence

\[
\sqrt{\alpha_n} = \sqrt{(p_1^2 + 1)} - p_1.
\]

Using (8.6) in (8.1) we obtain (8.4).

**Examples:**

Let \( n = 1 \). Since \( G_1 = 1 \), from Theorem 8.1 we have

\[
A \left( e^{-\pi} \right) = \frac{1}{2} e^{\pi \left( \frac{2^{1/4} - 1}{2^{1/4} + 1} \right)}.
\]

Let \( n = 2 \). Since \( g_2 = 1 \), from Theorem 8.1 we have

\[
A \left( e^{-\pi/\sqrt{2}} \right) = \frac{1}{2} e^{\pi \sqrt{2} \left( \frac{1 - (2 \sqrt{2} - 2)^{1/4}}{1 + (2 \sqrt{2} - 2)^{1/4}} \right)}.
\]

Let \( n = 3 \). Since \( G_3^{12} = 2 \), then Theorem 8.1 we have
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\[ A \left( e^{-\pi/\sqrt{3}} \right) = \frac{1}{2} e^{\pi \sqrt{3}} \frac{\left( (8+4\sqrt{3})^{1/4} - (7+4\sqrt{3})^{1/4} \right)}{\left( (8+4\sqrt{3})^{1/4} + (7+4\sqrt{3})^{1/4} \right)}. \]

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