\( \mu - \sigma \) PRE OPEN EQUIVALENT GENERALIZED TOPOLOGICAL SPACES

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ABSTRACT

In this paper we introduce and investigate \( \mu - \sigma \) Pre open equivalent generalized topologies. By using the family \( P(\mu) \) of all strong generalized topologies \( \mu \) and \( \sigma \) with \( \mu \sim \sigma \), we characterize the largest member of \( P(\mu) \).

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INTRODUCTION

Let \( X \) be a set. A subset \( \mu \) of \( \exp X \) called a generalized topology on \( X \) and \( (X, \mu) \) is called a generalized topological space \([2]\) if \( \mu \) has the following properties

(i) \( \varphi \in \mu \)

(ii) Any union of elements of \( \mu \) belongs to \( \mu \).

Let \( B \subseteq \exp X \) and \( \varphi \in B \). Then \( B \) is called a base \([2]\) for \( \mu \) if \( \{ \cup B; B \subseteq B \} = \mu \). We also say that \( \mu \) is generated by \( B \).

Generalized topological space is an important generalization of topological spaces and many interesting results have been obtained. In this paper, we investigate the equivalent of \( \text{pre open sets} \) in strong generalized topological spaces and give some characterizations of \( \text{pre open sets} \) and some relations among them.

Throughout this paper, a space \((X, \mu)\) or simply \(X\) for short, will always mean a strong generalized topological space with strong generalized topology \( \mu \) unless otherwise explicitly stated. Here a generalized topology \( \mu \) is said to be strong \([2]\) if \( X \in \mu \). Pre open in \((X, \mu)\) is called \( \mu \)-pre open.

In the generalized topology \( \mu \), the elements of \( \mu \) are called \( \mu \)-open sets. A subset \( A \) of \( X \) is said to be \( \mu \)-closed if \( X \setminus A \in \mu \).

That is, a subset \( A \) of \( X \) is called \( \mu \)-open (or \( \mu \)-closed) if \( A \in \mu \) (or \( X \setminus A \in \mu \)). For \( A \subseteq X \), let \( I(\mu) \) or \( I_\mu(\mu) \) be the largest \( \mu \)-open subset contained in \( A \). Equivalently, \( I(\mu) \) is the union of all \( \mu \)-open subsets of \( A \). Let \( C(\mu) \) or \( C_\mu(A) \) be the smallest \( \mu \)-closed subset containing \( A \). Equivalently \( C(\mu) \) is the intersection of all \( \mu \)-closed subsets which contain \( A \).

A point \( x \in X \) is called a \( \mu \)-cluster point of \( A \) if \( U \cap (A \setminus \{x\}) \neq \varnothing \) for each \( U \subseteq \mu \) with \( x \in \mu \). The set of all \( \mu \)-cluster points of \( A \) is denoted by \( d(\mu) \).

If \( \gamma: \rho(X) \rightarrow \rho(X) \) is a monotonic function defined on a non empty set \( X \) and \( \mu = \{A / A \subseteq \gamma(\mu)\} \), the family of all \( \gamma \)-open sets is also a generalized topology \([3]\).

The family of all monotonic functions defined on \( X \) is denoted by \( \Gamma \).

**Definition 1.1** \([4]\): A subset \( A \) of a space \((X, \mu)\) is said to be \( \alpha \)-open (resp., \( \text{semi open} \), \( \text{pre open} \), \( \beta \)-open \([6]\), \( \beta \)-closed \([6]\)) if

\[ A \in I_\alpha(C_\alpha(A)) \] (resp., \( A \in I_\beta(C_\beta(A)) \)).

We will denote the family of all \( \alpha \)-open sets by \( \alpha \) or \( \mu^\alpha \). The family of all \( \text{semi open sets} \) by \( \text{SO} (X, \mu) \). The family of all \( \text{pre open sets} \) or \( \text{PO} (X, \mu) \). The family of all \( \beta \)-open sets by \( \beta \) and family of all \( \beta \)-closed sets by \( \beta \).

If \((X, \mu)\) is a generalized topological space, then we say that a subset \( A \in I_\delta \subseteq \rho(X) \) \([7]\) if for every \( x \in A \), there exists a \( \mu \)-closed set \( Q \) such that \( x \in I_\mu(Q) \subseteq A \). Then \((X, \delta)\) is a generalized topological space \([7, \text{proposition 2.1}]\) such that \( \delta \subseteq \mu \) \([7, \text{Theorem 1}]\).

Elements of \( \delta \) are called the \( \delta \)-open sets of \((X, \mu)\). For \( A \subseteq X \),

\[ I_\delta(A) \] and \( C_\delta(A) \) are the interior and closure of \( A \) in \((X, \delta)\).

For a space \((X, \mu)\), the family \{ \( A \subseteq X / A \cap B \subseteq P(0, X, \mu) \) whenever \( B \in P(0, X, \mu) \) \} will be denoted by \( \mu^\gamma \).

**Remark 1.1** If \((X, \mu)\) is any generalized topological space which is not strong, then \( X \in \sigma \) and so it follows that always \( x \in b \) and \( x \in B \). In general, if \( X \notin \mu \), then \( X \notin \lambda \) for \( \lambda \in \{\mu, \delta, \alpha, \pi\} \). For example, let \( X \) be the set of all real numbers and \( \mu = \{\varnothing, \{0\}\} \). Then \( X \notin \lambda \) where \( \lambda \in \{\mu, \delta, \alpha, \pi\} \).

**Notation 1.1** Let \( X \) be a space. We use the following notation for \( x \in X \) and \( F \subseteq \exp X \)

(i) \( \cap F \neq \emptyset \)

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Proposition 1.1: Let $A$ be a subset of a generalized topological space $X$. Then

(i) $I_{\mu}(A) \subset A \subset C_{\mu}(A)$

(ii) $I_{\mu}(I_{\mu}(A)) = I_{\mu}(A)$

(iii) $C_{\mu}(C_{\mu}(A)) = C_{\mu}(A)$

(iv) $I_{\mu}(A) = A$ if and only if $A$ is open

(v) $C_{\mu}(A) = A$ if and only if $A$ is $\mu$-closed

(vi) If $A' \subset A$ then $I_{\mu}(A') \subset I_{\mu}(A)$

(vii) If $A' \subset A$ then $C_{\mu}(A') \supset C_{\mu}(A)$

(viii) If $A' \subset A$ then $d(A') \subset d(A)$

(ix) $C_{\mu}(X - A) = X - C_{\mu}(A)$

(x) $x \in C_{\mu}(A)$ if and only if $U \cap A \neq \emptyset$ for each $U \in \mu_x$

(xi) $x \in I_{\mu}(A)$ if and only if $U \subset A$ for some $U \in \mu_x$

(xii) $C(A) = A \cup d(A)$

(xiii) $x \notin d(\{x\})$ for each $x \in X$

(i) By proposition 1.1, $I_{\mu}(A) \subset A \subset C_{\mu}(A)$

(ii) $I_{\mu}(I_{\mu}(A)) = I_{\mu}(A)$

(iii) $C_{\mu}(C_{\mu}(A)) = C_{\mu}(A)$

(iv) $I_{\mu}(A) = A$ if and only if $A$ is open

(v) $C_{\mu}(A) = A$ if and only if $A$ is $\mu$-closed

Results 1.1

(i) $\mu(A \cap B) = \mu(A) \cap \mu(B)$ and $\mu(A \cup B) = \mu(A) \cup \mu(B)$

For, let $X = \{a, b, c\}$ and $\mu = \{\emptyset, X, \{a, b\}, \{b, c\}, \{a, c\}\}$

Take $A_1 = \{a, b\}$, $A_2 = \{b, c\}$ and $B_1 = \{a\}$, $B_2 = \{b\}$.

$x_1 = I_{\mu}(A_1)$ and $x_2 = I_{\mu}(A_2)$

Then $x_1 = I_{\mu}(A_1 \cap A_2) = I_{\mu}(\{a, b\} \cap \{b, c\}) = I_{\mu}(\{b\}) = \emptyset$

But $I_{\mu}(A_1) \cap I_{\mu}(A_2) = I_{\mu}(\{a, b\}) \cap I_{\mu}(\{b, c\}) = \emptyset$

Thus $\mu = \mu(A_1 \cup A_2) = \mu(A_1) \cup \mu(A_2)$

Definition 1.2

Let $X$ be a space.

(i) $x \in X$ and $U \in \mu_x$. Then $x$ is called representative element of $U$ if $U \supset x$ for each $V \in \mu_x$

(ii) $x \in X$ is called $C_{\mu}$-space if $C_{\mu}(x)$, Where $\mu$ is the set of all representative elements of sets of $\mu$

(iii) Let $x \in X$. The set $M_d(x) = \{U \in \mu_x : U \supset V \in \mu_x \Rightarrow V = U\}$ is called the minimal description of $x$

Remark 1.2

Let $x \in X$ and $x \notin C_{\mu}$ if and only if $|M_d(x)| > 1$ in which $|M_d(x)|$

Is the cardinality of $M_d(x)$

The following proposition shows that the equality of the above results 1.1 hold by using the sufficient condition of definition 1.2

Proposition 1.2

Let $A_1$ and $A_2$ be subsets of a $C_{\mu}$-space $X$.

(i) $I_{\mu}(A_1 \cap A_2) = I_{\mu}(A_1) \cap I_{\mu}(A_2)$

(ii) $C_{\mu}(A_1 \cup A_2) = C_{\mu}(A_1) \cup C_{\mu}(A_2)$

Proof:

(i) By proposition 1.1, $I_{\mu}(A_1 \cap A_2) \subset I_{\mu}(A_1) \cap I_{\mu}(A_2)$

If $x \in I_{\mu}(A_1) \cap I_{\mu}(A_2)$, then there are $U_1, U_2 \in \mu_x$ such that $U_1 \subset A_1$ and $U_2 \subset A_2$. Since $X$ is a $C_{\mu}$-space, $x \in C_{\mu}$.

Consequently, $x \in U_1 \subset U_1 \cup U_2 \subset A_1 \cup A_2$.

(ii) $\mu = \mu_1 \cup \mu_2$- separation if $x \in U \in \mu_x$, then $C_{\mu}(\{x\}) \in \mu_x$

Definition 2.4

A space $X$ is $\mu$-semi - $TD$ if and only if $\mu = \mu_1 \mu_2$. Consequently, if $(X, \mu) = \mu_1 \cup \mu_2$ then $\mu$ has a
largest member, namely $\mu^a = \mu^v$. The converse need not be true.

**Example 2.1:** Let $X = \{a, b, c\}$ and let $\mu = \{\emptyset, X, \{a, b\}\}$ then

$\mu = \mu^a$ and $(X, \mu)$ fails to be $\mu$ semi-TD. However, that $\mu^a$ is the largest member of $P(\mu)$.

**Theorem 2.1** Let $X$ be a space and let $\mu$ and $\sigma$ be two strong generalized topologies on $X$ satisfying $\mu \subseteq \sigma \subseteq \mu^a$.

Then the following are equivalent.

(i) $\mu \subseteq \sigma$

(ii) $C_\sigma(D) = \emptyset$ for each $\mu$ - dense set $D \subseteq X$

(iii) $I_\mu(A)$ is $\sigma$ - closed for each $\mu$ - codense set $A \subseteq X$

(iv) $I_\mu(C_\mu(A)) \subseteq C_\sigma(A)$ for every subset $A \subseteq X$.

(v) $I_\mu(C_\mu(A)) \cap (X - C_\mu(A)) \subseteq C_\sigma(X, \sigma)$ for every subset $A \subseteq X$.

**Proof:**

(i) $\Rightarrow$ (ii) Let $D \subseteq X$ be $\mu$ - dense and let $x \in C_\mu(D)$.

Since $D \subseteq X$ be $\mu$ - dense and let $x \in C_\mu(D)$ since $D \subseteq X$ be $\mu$ - dense and let $x \in C_\mu(D)$.

Thus $C_\sigma(D) \subseteq \sigma$.

(i) $\Rightarrow$ (i) By proposition 2.1(vi), it remains to show that $PO(X, \mu) \subseteq PO(X, \sigma)$.

Let $A \in PO(X, \mu)$. That is $A = U \cap D$ with $U \in \mu$ and $D$, the $\mu$ - dense.

Then $A \subseteq U \cap C_\mu(D) \subseteq \sigma$.

Since $U \cap C_\mu(D) \subseteq C_\mu(U \cap D) = C_\mu(A)$

we clearly have $PO(X, \mu) \subseteq PO(X, \sigma)$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv) Let $A \subseteq X$. Since $A \cup (X - C_\mu(A))$ is $\mu$ - dense we have $C_\mu(A) \cup C_\mu(X - C_\mu(A)) \subseteq \sigma$.

Hence, $I_\mu(C_\mu(A)) \cap (C_\mu(A) \cup C_\mu(X - C_\mu(A))) = I_\mu(C_\mu(A)) \subseteq \sigma$.

(iv) $\Rightarrow$ (v) Let $A \subseteq X$ and take $B = I_\mu(C_\mu(A)) \cap (X - C_\mu(A))$.

Then clearly $B \subseteq \sigma$ and $X - B$ is $\mu$ - dense.

Hence, $I_\mu(C_\mu(X - B)) \cup C_\mu(X - B) = X - B \subseteq \sigma$.

And so $B \subseteq C_\mu(X, \sigma)$.

(v) $\Rightarrow$ (ii) is obvious.

**Lemma 2.2:** For a space $X$, let $A \subseteq \mu$ and let $D \subseteq X$ be $\mu$ - dense. Then $A \cap C_\sigma(A \cap D) \subseteq A \cap I_\mu(C_\mu(A \cap D))$.

**Proof:** Let $x \in A \cap C_\sigma(A \cap D)$.

Suppose $S = A \cap (D \cup \{x\})$. Then $S \subseteq PO(X, \mu)$ and $x \in S$.

Since $C_\mu(S) = C_\mu(A \cap D)$ we have $x \in A \cap I_\mu(C_\mu(A \cap D))$.

Thus $A \cap C_\sigma(A \cap D) \subseteq A \cap I_\mu(C_\mu(A \cap D))$.

**Theorem 2.2:** Let $X$ be a strong generalized topological space with strong generalized topology $\mu$ and $\sigma$ and let $A \subseteq C_\mu(X, \mu)$.

If $\sigma$ on $X$ having $\mu \cap (A \cap X)$ as a subbase then $\mu \subseteq \sigma$.

**Proof:** Clearly we have $\mu \subseteq \sigma \subseteq \mu^a$.

Let $D \subseteq X$ be $\mu$ - dense and let $x \in C_\mu(D)$.

Now, $A \cap C_\mu(D) = A \cap I_\mu(C_\mu(A \cap D))$ and $A \cap C_\mu(A \cap D)$ and $A \cap C_\mu(A \cap D)$.

Then $A \cap C_\sigma(A \cap D) = A \cap I_\mu(C_\mu(A \cap D))$.

**Corollary 2.1:** For a space $(X, \mu)$, suppose that $\mu$ is the largest member of $P(\mu)$.

Then $C_\mu(X, \mu^*) \subseteq \mu^a$.

In Remark 2.1, we have pointed out that $\sigma \subseteq \mu^a$ whenever $\mu \subseteq \sigma$.

For strengthening this, in the space $(X, \mu)$

Let $M_2 = \{x \in X \setminus \{x\} \in C_\mu(X, \mu^*)\}$. Let $\mu^*$ be the strong generalized topology on $X$ obtained from $\mu^a$ by making $MAN \mu - open$ discrete subspace. That is for $V \subseteq X$ we have $V \subseteq \mu^*$ if and only if $V = U \cup K$ for some $U \subseteq \mu$ and some $K \subseteq M_2$.

Clearly $\mu \subseteq \mu^* \subseteq \mu^v$.

**Lemma 2.3:** For a space $(X, \mu)$, let $A \subseteq C_\mu(X, \mu)$ and $I_\mu(A) = \emptyset$. Then $A \subseteq Mx$.

**Proof:** Let $x \in A$ be arbitrary. Since $\mu \subseteq \sigma, we$ have $\sigma \subseteq \mu^a$.

Then $W \cap C_\mu(I_\mu(W)) \subseteq \mu^a$.

Suppose that $K = W \cap (X - C_\mu(I_\mu(W)))$.

Then $K \subseteq \sigma$ and $K \subseteq \mu - condense$.

By Theorem 2.1(iii), $K \subseteq C_\sigma(X, \sigma)$.

Hence $K \subseteq C_\mu(X, \mu^*)$.

Our claim follows now from lemma 2.3.

**Theorem 2.4:** For a space $(X, \mu)$, suppose that $P(\mu)$ has a largest member, say $\sigma$. Then $\sigma \subseteq \mu^*$.

**Proof:** By Theorem 2.3, $\sigma \subseteq \mu^*$. Since $\sigma - \mu$ we have $\sigma \subseteq \mu^a$ and consequently $C_\mu(X, \sigma^a) \subseteq \mu^a$ by corollary 2.1.

Hence, $K \subseteq \sigma$ for each subset $K \subseteq \mu^a$.

Clearly, $\mu^a \subseteq \sigma$ and so $\sigma \subseteq \mu^*$.

**References:**


