INTUITIONISTIC FUZZY PRE SEMI EXTREMALLY DISCONNECTED SPACES

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ABSTRACT

In this paper, a new class of intuitionistic fuzzy topological spaces called intuitionistic fuzzy pre semi extremally disconnected spaces is introduced and several other properties are discussed.

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INTRODUCTION

After the introduction of the concept of fuzzy sets by Zadeh [17], several researches were conducted on the generalizations of the notion of fuzzy set. The concept of "Intuitionistic fuzzy sets" was first published by Atanassov [2] and many works by the same author and his colleagues appeared in the literature [3-5]. Later this concept was generalized to "Intuitionistic L-fuzzy sets" by Atanassov and stoeva [6]. An introduction to intuitionistic fuzzy topological space was introduced by Dogan Coker [9]. Several types of fuzzy connectedness in intuitionistic fuzzy topological spaces defined by Coker (1997). The construction is based on the idea of intuitionistic fuzzy set developed by Atanassov (1983, 1986; Atanassov and Stoeva, 1983). In this paper a new class of intuitionistic fuzzy topological spaces namely, intuitionistic fuzzy pre semi extremally disconnected spaces is introduced by using the concepts of fuzzy extremally disconnected spaces [7], fuzzy pre open sets [8]. 'Intuitionistic fuzzy pre semi closed sets' was introduced by [1]. Tietze extension theorem for intuitionistic fuzzy pre semi extremally disconnected spaces has been discussed in [15]. Some interesting properties and characterizations are studied.

2. Preliminaries

Definition 2.1[4] Let $X$ be a non empty fixed set. An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$ where the functions $\mu_A : X \rightarrow I$ and $\gamma_A : X \rightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.2[4] Let $X$ be a non empty set and the IFSs $A$ and $B$ be in the form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$, $B = \{ (x, \mu_B(x), \gamma_B(x)) : x \in X \}$. Then

(a) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(c) $\overline{A} = \{ (x, \gamma_A(x), \mu_A(x)) : x \in X \}$;

(d) $A \cap B = \{ (x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) : x \in X \}$;

(e) $A \cup B = \{ (x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x)) : x \in X \}$;

(f) $\cap A = \{ (x, \mu_A(x), 1 - \mu_A(x)) : x \in X \}$;

(g) $\cup A = \{ (x, 1 - \gamma_A(x), \gamma_A(x)) : x \in X \}$

Definition 2.3[9] Let $X$ be a non empty set and let $\{ A_i : i \in J \}$ be an arbitrary family of IFSs in $X$. Then (a) $\bigcap A = \{ (x, \land \mu_A(x), \lor \gamma_A(x)) : x \in X \}$;

(b) $\bigcup A = \{ (x, \lor \mu_A(x), \land \gamma_A(x)) : x \in X \}$

Definition 2.4[9] Let $X$ be a non empty fixed set. Then, $0_+ = \{ (x, 0, 1) : x \in X \}$ and $1_- = \{ (x, 1, 0) : x \in X \}$.

Remark 2.1[4] For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A \rangle$.
Definition 2.5.[9] Let $X$ and $Y$ be two non empty fixed sets and $f : X \rightarrow Y$ be a function. Then
$$
\text{(a) If } B = \{ \{ y, \mu_B(y), \gamma_B(y) \} : y \in Y \} \text{ is an } IFS \text{ in } Y, \text{ then the pre image of } B \text{ under } f, \text{ denoted by } f^{-1}(B), \text{ is the } IFS \text{ in } X \text{ defined by } f^{-1}(B) = \{ (x, f^{-1}(\mu_B(x)), f^{-1}(\gamma_B(x))) : x \in X \}.
$$
$$
\text{(b) If } A = \{ (x, \lambda_A(x), \nu_A(x)) : x \in X \} \text{ is an } IFS \text{ in } X, \text{ then the image of } A \text{ under } f, \text{ denoted by } f(A), \text{ is the } IFS \text{ in } Y \text{ defined by } f(A) = \{ (y, f(\lambda_A(x)), (1 - f(1 - \nu_A(x)))) : y \in Y \} \text{ where,}
$$
$$
f(\lambda_A(x)) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
0 & \text{otherwise},
\end{cases}
$$
$$
(1 - f(1 - \nu_A(x)))(y) = \begin{cases} 
\inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\
1 & \text{otherwise}.
\end{cases}
$$
for the $IFS$ $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}.$

Definition 2.6.[9] Let $X$ be a non empty set. An intuitionistic fuzzy topology (IFT for short) on a non empty set $X$ is a family $\tau$ of intuitionistic fuzzy sets (IFSs for short) in $X$ satisfying the following axioms: (T1) $\emptyset, 1 \in \tau,$ (T2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau,$ (T3) $\bigcup_{i \in I} G_i \in \tau$ for any arbitrary family $\{G_i : i \in I\}.$

In this case the pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS for short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS for short) in $X.$

Definition 2.7.[9] Let $X$ be a non empty set. The complement $A$ of an IFOS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS for short) in $X.$

Definition 2.8.[9] Let $(X, \tau)$ be an IFTS and $A = \{ (x, \lambda_A(x), \gamma_A(x)) \}$ be an IFS in $X.$ Then the fuzzy interior and fuzzy closure of $A$ are defined by
$$
cl(A) = \bigcap \{ K : K \text{ is an IFS in } X \text{ and } A \subseteq K \},
$$
$$
int(A) = \bigcup \{ G : G \text{ is an IFS in } X \text{ and } G \subseteq A \}.
$$

Remark 2.2.[9] Let $(X, \tau)$ be an IFTS. $cl(A)$ is an IFCS and $int(A)$ is an IFOS in $X,$ and $A$ is an IFOS in $X$ iff $cl(A) = A;$ $A$ is an IFCS in $X$ iff $int(A) = A.$

Proposition 2.1.[9] Let $(X, \tau)$ be an IFTS. For any IFS $A$ in $(X, \tau),$ we have
$$
\text{(a) } cl(A) = \overline{int(A)}, \quad \text{(b) } int(A) = \overline{cl(A)}.
$$

Definition 2.9.[9] Let $(X, \tau)$ and $(Y, \phi)$ be two IFTSs and let $f : X \rightarrow Y$ be a function. Then $f$ is said to be fuzzy continuous iff the pre image of each IFS in $\phi$ is an IFS in $\tau.$

Definition 2.10.[12] Let $(X, \tau)$ and $(Y, \phi)$ be two IFTSs and let $f : X \rightarrow Y$ be a function. Then $f$ is said to be fuzzy open(resp. closed) iff the image of each IFS in $\tau$ (resp. $(1 - \tau)$) is an IFS in $\phi$ (resp. $(1 - \phi)$).

An IFTS $(X, T)$ represent intuitionistic fuzzy topological spaces and for a subset $A$ of a space $(X, T),$ $IFcl(A), IFint(A), IFPScl(A), IFPSint(A),$ and $\overline{A}$ denote an intuitionistic fuzzy closure of $A,$ an intuitionistic fuzzy interior of $A,$ intuitionistic fuzzy pre semi closure of $A,$ an intuitionistic fuzzy pre semi interior of $A$ and the complement of $A$ in $X$ respectively.

Definition 2.11.[16] A subset $A$ of an IFTS $(X, T)$ is called an IF semi open pre semi open set if $A \subseteq IFcl(IF int(IFcl(A)))$ and an IF semi pre closed set if $IF int(IFcl(IF int(A))) \subseteq A.$

Definition 2.12.[14] A subset $A$ of an IFTS $(X, T)$ is called an IF generalized closed (briefly IF g-closed) set if $IF cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is IF open in $(X, T).$ The complement of an IF g-closed set is called an IF g-open set;

Definition 2.13.[1] A subset $A$ of an IFTS $(X, T)$ is called intuitionistic fuzzy pre semi closed(IF pre semi closed for short) if $IF spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is IF open in $(X, T).$

Definition 2.14.[1] A subset $A$ of an IFTS $(X, T)$ is called intuitionistic fuzzy pre semi open(IF pre semi open for short) if $A$ is IF pre semi closed.

Definition 2.15.[1] A function $f : (X, T) \rightarrow (Y, S)$ is called intuitionistic fuzzy pre semi continuous (IF pre semi continuous for short) if $f^{-1}(V)$ is an IF pre semi closed set of $(X, T)$ for every IF closed set $V$ of $(Y, S).$

3. Intuitionistic fuzzy pre semi extremely disconnected spaces

Definition 3.1. Let $(X, T)$ be an IFTS. Let $A$ be any IF pre semi open set in $(X, T).$ If $IF cl(A)$ is IF pre semi closed, then $(X, T)$ is said to be intuitionistic fuzzy pre semi extremely disconnected(for short IF pre semi extremely disconnected).

Proposition 3.1. For an IFTS $(X, T),$ the following are equivalent
$$
\text{(a) } (X, T) \text{ is IF pre semi extremely disconnected.}
$$
$$
\text{(b) For each IF pre semi closed set } A, \text{ IFPS int } A \text{ is IF pre semi closed.}
$$
$$
\text{(c) For each IF pre semi open set } A, \text{ we have } IFPScl(IFPS int(A)) = IFPScl(A).
$$
$$
\text{(d) For each pair of IF pre semi open sets } A \text{ and } B \text{ in } (X, T) \text{ with } IFPSclA = B, \text{ we have } IFPSclA = IFPSclB.
$$

Proof. (a) $\Rightarrow$ (b): Let $A$ be any IF pre semi closed, we claim that $IFPS int A$ is an IF pre semi closed. Now $A$ is IF pre semi open. So by assumption (a) $IFPScl\overline{A}$ is IF pre semi open. That is $IFPS int A$ is IF pre semi closed.
(b) \( \Rightarrow \) (c): Let \( A \) be \( IF \) pre semi open. Then \( IFPScl A = IFPS int A \). Consider
\[
IFPScl A + IFPScl(IFPScl A) = IFPScl(IFPS int A). \]
Since \( A \) is \( IF \) pre semi open, \( \overline{A} \) is \( IF \) pre semi closed and by (b) \( IFPS int \overline{A} \) is \( IF \) pre semi closed. Therefore,
\[
IFPScl(IFPS int \overline{A}) = IFPS int \overline{A}. \]
Now, \( IFPScl A + IFPScl(IFPScl A) = IFPScl(IFPS int \overline{A}) = IFPScl A + IFPScl A \).
Hence, \( IFPScl(IFPS(int A)) = IFPScl(A) \).

(c) \( \Rightarrow \) (d): Let \( A \) and \( B \) be any \( IF \) pre semi open sets such that \( IFPScl A = B \), i.e. \( B = IFPS int \overline{A} \).
By (c), \( IFPScl(IFPScl A) = IFPScl(A) \).

Therefore,
\[
IFPScl B = IFPScl(IFPScl A) = IFPScl(A). \]
From (3.1) and (3.2), we have\[
IFPScl(IFPS(int A)) = IFPScl(A). \]
But \( B = IFPS int \overline{A} \).
Therefore, \( IFPScl A = IFPScl B \).

(d) \( \Rightarrow \) (a): Let \( A \) be any \( IF \) pre semi open set. Let \( IFPScl A = B \). From (d), it follows that \( IFPScl A = IFPScl B \).
Hence, \( (X,T) \) is \( IF \) pre semi extremally disconnected space.

**Proposition 3.2.** Let \( (X,T) \) be an \( IFTS \). Then \( (X,T) \) is an \( IF \) pre semi extremally disconnected space if and only if for \( IF \) pre semi open set \( A \) and \( IF \) pre semi closed set \( B \) such that \( A \subseteq B \), we have \( IFPS cl A \subseteq IFPS int B \).

**Notation.** An \( IFS \) which is both \( IF \) pre semi open and \( IF \) pre semi closed is called \( IF \) pre semi clopen set.

**Remark 3.1.** Let \( (X,T) \) be an \( IF \) pre semi extremally disconnected space. Let \( \{ A_i, B_i \}_{i \in N} \) be an increasing collection of \( IF \) pre semi clopen sets of \( (X,T) \) such that \( IFPS cl A_i \subseteq B_i \), for all \( i, j \in N \). By Proposition 3.2, \( IFPS cl A_i \subseteq IFPS cl A \cap IFPS int B_i \). Put \( C = IFPS cl A \cap IFPS int B \) for all \( i, j \in N \). Now, \( C \) satisfies our required condition.

**Proposition 3.3.** Let \( (X,T) \) be an \( IF \) pre semi extremally disconnected space. Let \( \{ A_q \}_{q \in Q} \) and \( \{ B_q \}_{q \in Q} \) be the monotone increasing collections of \( IF \) pre semi open sets and \( IF \) pre semi closed sets of \( (X,T) \) respectively and suppose that \( A_q \subseteq B_q \) whenever \( q < r \) (\( Q \) is the set of rational numbers). Then there exists a monotone increasing collection \( \{ C_q \}_{q \in Q} \) of \( IF \) pre semi clopen sets of \( (X,T) \) such that \( IFPS cl A_q \subseteq C_q \), for all \( q \in Q \).

**4. Properties and characterizations of Intuitionistic fuzzy pre semi extremally disconnected spaces**

In this section various properties and characterizations of intuitionistic fuzzy pre semi extremally disconnected spaces are discussed.

**Definition 4.1.** Let \( (X,T) \) be an \( IFTS \). A mapping \( f : X \rightarrow R(I) \) is called lower (resp. upper) \( IF \) pre semi continuous, if \( f^{-1}(R) \) (resp. \( f^{-1}(L) \)) is an \( IF \) pre semi open set (resp. \( IF \) pre semi open/ \( IF \) pre semi closed) for each \( t \in R \).

**Proposition 4.1.** Let \( (X,T) \) be an \( IFTS \). Let \( A \in I^X \), and let \( f : X \rightarrow R(I) \) be such that \( f(x)(t) = \begin{cases} 1 & \text{if } \ t < 0, \\ A(x) & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t > 1, \end{cases} \)
for all \( x \in X \). Then \( f \) is lower (resp. upper) \( IF \) pre semi continuous iff \( A \) is \( IF \) pre semi open (resp. \( IF \) pre semi open/ \( IF \) pre semi closed) set.

**Definition 4.2.** Let \( (X,T) \) be an \( IFTS \). The characteristic function of \( IF \) \( A \) in \( X \) is the function \( \chi_A : X \rightarrow I(L) \) defined by \( \chi_A(x) = (A(x), (x \in X)). \)

**Proposition 4.2.** Let \( (X,T) \) be an \( IFTS \), and let \( A \in I^X \). Then \( \chi_A \) is lower (resp. upper) \( IF \) pre semi continuous iff \( A \) is \( IF \) pre semi open (resp. \( IF \) pre semi open/ \( IF \) pre semi closed).

**Proof.** The proof follows from Proposition 4.1.

**Definition 4.3.** Let \( (X,T) \) and \( (Y,S) \) be \( IFTSs \). A mapping \( f : (X,T) \rightarrow (Y,S) \) is called intuitionistic fuzzy strong pre semi continuous (for short \( IF \) strong pre semi continuous) if \( f^{-1}(A) \) is \( IF \) pre semi clopen in \( (X,T) \) for every \( IF \) pre semi open set in \( (Y,S) \).

**Proposition 4.3.** Let \( (X,T) \) be an \( IFTS \). Then the following are equivalent.
(a) \( (X,T) \) is \( IF \) pre semi extremally disconnected,
(b) \( g, h : X \rightarrow R(I) \), \( g \) is lower \( IF \) pre semi continuous, \( h \) is upper \( IF \) pre semi continuous and \( g \subseteq h \), then there exists an \( IF \) strong pre semi continuous function \( f : (X,T) \rightarrow R(I) \) such that \( g \subseteq f \subseteq h \).
(c) If \( A \) and \( B \) are \( IF \) pre semi open sets such that \( B \subseteq A \), then there exists an \( IF \) strong pre semi continuous function \( f : (X,T) \rightarrow [0,1] \) such that \( B \subseteq \overline{L} f \subseteq R_0 \) for all \( r \in Q \).

**Proof.** (a) \( \Rightarrow \) (b): Define \( H_r = L_r h \) and \( G_r = (R_r) g \), \( r \in Q \). Thus we have two monotone increasing families respectively \( IF \) pre semi open and \( IF \) pre semi closed sets of \( (X,T) \). Moreover \( H_r \subseteq G_r \) if \( r < s \). By Proposition 3.3,
there exists a monotone increasing family \( \{ F_r \}_{r \in Q} \) of IF pre semi clopen sets of \((X, T)\) such that 
\[
\text{IFPScl}(H_r) \subseteq F_r
\]
and \( F_r \subseteq \text{IFPS int}(G_s) \) whenever \( r < s \). Let \( V_t = \bigcap_{r < t} \overline{F_r} \) for all \( t \in R \), we define a monotone decreasing family \( \{ V_t / t \in R \} \subseteq I^X \) Moreover we have \( \text{IFPScl}(V_t) \subseteq \text{IFPS int}(V_s) \) whenever \( s < t \).
We have 
\[
\bigcup_{t \in R} V_t = \bigcup_{t \in R} \cap_{r < t} \overline{F_r} = \bigcup_{t \in R} \cap_{r < t} (G_r) = \bigcup_{t \in R} \cap_{r < t} g^{-1}(R_t) = \bigcup_{t \in R} g^{-1}(R_t) = g^{-1} \left( \bigcup_{t \in R} (R_t) \right) = 1.
\]

Similarly, \( \cap_{t \in R} V_t = 0 \).

We now define a function \( f : X \rightarrow R(I) \) satisfying the required properties. Let \( f(x)(i) = V_t(x) \) for all \( x \in X \) and \( t \in R \). By the above discussion, it follows that \( f \) is well defined. To prove \( f \) is IF strong pre semi continuous, we observe that \( \cap_{s > t} V_s = \cap_{s > t} \text{IFPS int}(V_s) \) and 
\[
\cap_{s < t} V_s = \cap_{s < t} \text{IFPScl}(V_s).
\]

Then \( f^{-1}(R_t) = \cap_{s > t} V_s \) is IF pre semi open. And 
\[
f^{-1}(\overline{L_t}) = \cap_{s < t} V_s = \cap_{s < t} \text{IFPScl}(V_s)
\]
is IF pre semi closed. Therefore \( f \) is IF strong pre semi continuous. To conclude the proof it remains to show that \( g \leq f \leq h \), that is 
\[
g^{-1}(\overline{L_t}) \subseteq f^{-1}(\overline{L_t}) \subseteq h^{-1}(\overline{L_t})
\]
and 
\[
g^{-1}(R_t) \subseteq f^{-1}(R_t) \subseteq h^{-1}(R_t)
\]
for each \( t \in R \).

We have, 
\[
g^{-1}(\overline{L_t}) = \cap_{s < t} g^{-1}(\overline{L_t}) = \cap_{s < t} \cap_{r < s} (G_r) \subseteq \cap_{s < t} \cap_{r < s} (\overline{F_r}) = \cap_{s < t} V_s = f^{-1}(\overline{L_t})
\]
And, 
\[
f^{-1}(\overline{L_t}) = \cap_{s < t} V_s = \cap_{s < t} \cap_{r < s} (\overline{F_r}) \subseteq \cap_{s < t} \cap_{r < s} (G_r) \subseteq \cap_{s < t} h^{-1}(\overline{L_t}) = h^{-1}(\overline{L_t})
\]
Similarly, 
\[
g^{-1}(R_t) = \cup_{s > t} g^{-1}(R_r) = \cup_{s > t} \cup_{r > s} (G_r) \subseteq \cup_{s > t} \cup_{r > s} (\overline{F_r}) \cup_{s > t} V_s = f^{-1}(R_t)
\]
And, 
\[
f^{-1}(R_t) = \cup_{s > t} V_s \cup_{s > t} \cup_{r > s} (\overline{F_r}) \subseteq \cup_{s > t} \cup_{r > s} (H_r) = \cup_{s > t} \cup_{r > s} h^{-1}(\overline{L_t}) = h^{-1}(R_t) = h^{-1}(R_t)
\]
Thus \( b \Rightarrow (c) \) is proved.

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