PGPRD-SETS AND ASSOCIATED SEPARATION AXIOMS

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INTRODUCTION

Tong [16] introduced the notion of D-sets by using open sets and used this notion to define some separation axioms. The topologists Caldas et al. [4,5]; Jafari [9]; Balasubramanian [2,3] defined the concepts of s-D₀ α-D₀, p-D₀, g-D₀ and gpr-D₀ (i=0, 1, 2) spaces respectively. In this paper, we introduce the notion of pgpr-D-sets as the difference of two pgpr-open sets and investigate their basic properties. Using pgpr-D-sets we introduce and study pgpr-D₀, s-D₀, α-D₀, g-D₀ and gpr-D₀ (i=0, 1, 2) spaces.

2. Preliminaries

Throughout this paper (X,τ) denotes a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of X, cl(A) and int(A) respectively denote the closure of A and the interior of A in (X,τ). We recall the following definitions and results that will be useful. A subset A of (X,τ) is regular open [15] if A=cl(int(A)); regular closed if A=int(cl(A)); pre-open [12] if A⊆cl(int(A)); pre-closed if int(cl(A))⊆A; semi-open [10] if A⊆int(cl(A)); semi-closed if int(cl(A))⊆A; α-open [13] if A⊆cl(int(cl(A))); α-closed if cl(int(cl(A)))⊆A. The pre-closure of A is the intersection of all pre-closed sets containing A and is denoted by pcl(A). Again a subset B of (X,τ) is called g-closed [11] (resp. rg-closed [14]) if cl(B)⊆U whenever B⊆U and U is open (resp. regular open) in X; gpr-closed [8] and pgpr-closed [1] are analogously defined by taking pre-closure instead of closure and regular open, rg-open sets respectively, instead of open set in X. The complement of g-closed set is g-open. The rg-open, gpr-open and pgpr-open sets can be analogously defined. The pgpr-closure of B is the intersection of all pgpr-closed sets containing B and is denoted by pgpr-cl(B).

Notations: The collection of all pre-open and pgpr-open sets in (X,τ) is denoted by PO(X,τ) and PGPRO(X,τ) respectively.

Definition 2.1: A subset A of a space X is called a D-set [16] (resp. pD-set[9], αD-set[5], sD-set[4], gD-set[2] and gprD-set[3]) if there are two open (resp. pre-open, α-open, semi-open, g-open and gpr-open) sets U and V such that U∩X=A∪U\V.

Definition 2.2: A space (X,τ) is said to be pgpr-T₀[7] if for any two distinct points x and y of X, there exist a pgpr-open set G such that (xεG and y∉G) or (yεG and x∉G); pgpr-T₁[7] if for any two distinct points x and y of X, there exist pgpr-open sets G and H such that xεG but y∉G and yεH but x∉H; pgpr-T₂[7] if for x ≠ yεX, there exist disjoint pgpr-open sets G and H such that xεG and yεH.

Definition 2.3: A space (X,τ) is said to be D₁[16] (resp. pD₁[9], αD₁[5], sD₁[4] and gprD₁[3]) if for any two distinct points x and y of X, there exist D-sets (resp. pD-sets, αD-sets, sD-sets and gprD-sets) G and H such that xεG but y∉G and yεH but x∉H.

Definition 2.4: A space (X,τ) is said to be D₂[16] (resp. pD₂[9], αD₂[5], sD₂[4] and gprD₂[3]) if for any two distinct points x and y of X, there exist disjoint D-sets (resp. pD-sets, αD-sets, sD-sets and gprD-sets) G and H such that xεG and yεH.

Definition 2.5: A function f: X→Y is pgpr-irresolute[1] if f⁻¹(V) is pgpr-closed in X for every pgpr-closed set V of Y.
3. pgpr-D-sets
In this section, we introduce pgpr-D-sets and investigate their basic properties.

**Definition 3.1:** A subset A of a space X is called a pgpr-D-set if there are two pgpr-open sets U and V such that U≠X and A=U∩V.

Every proper pgpr-open set is a pgpr-D-set. However, the converse is not true.

**Proposition 3.2:** (i) Every pD-set is a pgpr-D-set (ii) Every pgpr-D-set is a gpr-D-set.

**Diagram 3.3:** Definition 3.4: A topological space (X,τ) is said to be pgpr-Dο if for any two distinct points x and y of X, there exists a pgpr-D-set G such that x∈G and y∉G or y∉G and x∉G.

**Theorem 3.5:** Every topological space is pgpr-Dο.

**Proof:** Let (X,τ) be a topological space. Then by Proposition 3.7 of [7], (X,τ) is pgpr-Tο. Since (X,τ) is pgpr-Tο for any two distinct points x and y of X, there exists a pgpr-open set G such that x∈G and y∉G or y∉G and x∉G. Since either y∉G or x∉G, we have G≠X. That is G is a proper pgpr-open set and hence G is a pgpr-D-set. This proves that (X,τ) is pgpr-Dο.

4. pgpr-D1 spaces
In section 3, we have proved that every topological space is pgpr-Dο. Therefore, it is worthy to define pgpr-D1 spaces.

**Definition 4.1:** A topological space (X,τ) is said to be pgpr-D1 if for any two distinct points x and y of X, there exist pgpr-D-sets G and H such that x∈G but y∉G and y∉H but x∉H.

**Proposition 4.2:** Every p-D1 space is a pgpr-D1 space.

**Proposition 4.3:** Every topological space is gpr-D1.

**Proof:** Let (X,τ) be a topological space. By using Theorem 3.13 of [6], (x) is gpr-open for every x∈X. Since every proper gpr-open set is a gpr-D-set, (x) is a gpr-D-set for every x∈X. Then for any two distinct points x and y of X, x∈(x) but y∉(x) and y∉(y) but x∉(y). By Definition 2.3, (X,τ) is gpr-D1.

From the above discussions, we have the following diagram.

**Diagram 4.4:**

\[ PO(X,τ) = PGPRO(X,τ) \] holds for some topological spaces. However, there are topological spaces for which PO(X,τ)≠PGPRO(X,τ). After examining the topologies on 3 elements, 4 elements and 5 elements we make the following conjecture.

**Conjecture 4.5:** If (X,τ) is a topological space, then exactly one of the following holds:

(i) (X,τ) is both p-D1 and pgpr-D1.
(ii) (X,τ) is neither p-D1 nor pgpr-D1.

**Proposition 4.6:** Every pgpr-T1 space is pgpr-D1.

**Definition 4.7:** A topological space (X,τ) is pgpr-symmetric if for x and y in X, x∈pgpr-cl(y) implies y∈pgpr-cl(x).

**Theorem 4.8:** If a topological space (X,τ) is pgpr-symmetric then it is pgpr-T1.

**Proof:** Let (X,τ) be pgpr-symmetric. By Proposition 3.7 of [7], (X,τ) is pgpr-Tο. Let x and y be distinct points of X. Since (X,τ) is pgpr-Tο, we may assume that x∈G⊆X\{y} for some pgpr-open set G. Then x∉pgpr-cl(y). Since (X,τ) is pgpr-symmetric, y∉pgpr-cl(x). That is, there exists a pgpr-open set G such that x∈G⊆X\{x}. Thus (X,τ) is pgpr-T1.

**Corollary 4.9:** Let (X,τ) be pgpr-symmetric. Then (X,τ) is pgpr-D1 if and only if it is pgpr-T1.

**Theorem 4.10:** If f : (X,τ) → (Y,σ) is a pgpr-irresolute, surjective function and E is a pgpr-D-set in Y, then the inverse image of E is a pgpr-D-set in X.

**Proof:** Suppose f : (X,τ) → (Y,σ) is pgpr-irresolute and surjective. Let E be a pgpr-D-set in Y. Then there are pgpr-open sets U and V in Y such that E=U\∩V and U≠∅. Since f is pgpr-irresolute, f^{-1}(U_1) and f^{-1}(U_2) are pgpr-open in X. Since U_1≠Y and f is surjective, f^{-1}(U_1)≠X. Hence f^{-1}(E) = f^{-1}(U_1) \∩ f^{-1}(U_2), which is a pgpr-D-set in X.

**Theorem 4.11:** Let f : (X,τ) → (Y,σ) be pgpr-irresolute and bijective. If (Y,σ) is pgpr-D1, then (X,τ) is also pgpr-D1.

**Proof:** Suppose f : (X,τ) → (Y,σ) is pgpr-irresolute and bijective. Let (Y,σ) be pgpr-D1 and x, y any pair of distinct points in X. Since f is injective and Y is pgpr-D1, there exist pgpr-D-sets G_x and G_y of Y containing f(x) and f(y) respectively, such that f(x)∉G_y and f(y)∉G_x. By Theorem 4.10, f^{-1}(G_x) and f^{-1}(G_y) are pgpr-D-sets in X containing x and y respectively. This implies that (X,τ) is a pgpr-D1 space.

**Theorem 4.12:** A topological space (X,τ) is pgpr-D if and only if for each pair of distinct points x, y∈X, there exists a pgpr-irresolute, surjective function f : (X,τ) → (Y,σ), where (Y,σ) is pgpr-D1 such that f(x)≠f(y) distinct.

**Proof:** Necessity: For every pair of distinct points of X, it suffices to take the identity function on X. Sufficiently: Let x and y be any pair of distinct points of X. By assumption, there exists a pgpr-irresolute, surjective function f from a space (X,τ) onto an pgpr-D1 space (Y,σ), such that f(x)≠f(y). Therefore, there exist disjoint pgpr-D-sets G_x and G_y in Y such that f(x)∉G_y and f(y)∉G_x. Since f is pgpr-irresolute and surjective, by Theorem 4.10, f^{-1}(G_x) and f^{-1}(G_y) are disjoint pgpr-D-sets in X containing x and y respectively. Therefore (X,τ) is a pgpr-D1 space.

5. pgpr-D2 spaces
In this section, we introduce pgpr-D2 spaces and obtain some of their basic properties.

**Definition 5.1:** A topological space (X,τ) is said to be pgpr-D2 if for x ≠ y∈X, there exist disjoint pgpr-D-sets G and H such that x∈G and y∈H.

**Proposition 5.2:** Every p-D2 space is a pgpr-D2 space.

**Proposition 5.3:** Every topological space is gpr-D2.

**Diagram 5.4:**
H such that x ∈ G and y ∈ H. Since y ∈ G and x ∈ H, G and H are different from X; So G and H are disjoint pgprD-sets. Thus (X, τ) is pgpr-D2.

**Proposition 5.7:** Every pgpr-D2 space is pgpr-D0.

**Proof:** Suppose (X, τ) is pgpr-D2. Then by Definition 5.1, for any two distinct points x and y of X, there exist disjoint pgprD sets G and H such that x ∈ G and y ∈ H. Clearly x ∈ G but y ∉ G and y ∈ H but x ∉ H. It follows that (X, τ) is pgpr-D1.

**Theorem 5.8:** A topological space (X, τ) is pgpr-D1 if and only if it is pgpr-D2.

**Proof:** The sufficiency follows from Proposition 5.7. To prove the necessity, let (X, τ) be pgpr-D1. Then for each distinct pair x, y ∈ X, we have pgprD-sets G1 and G2 such that x ∈ G1, y ∉ G1, and y ∈ G2, x ∉ G2. Let G1 = U1 \ U2, G2 = U3 \ U4. Then x ∈ G1 = U1 \ U2 ⇒ x ∈ U1 and x ∉ U2. Since y ∉ G1 = U1 \ U2 we have two sub cases: (a) y ∉ U1 or (b) y ∈ U1 and y ∈ U2. Now y ∈ G2 = U3 \ U4 ⇒ y ∈ U3 and y ∈ U4. Since x ∉ G2 = U3 \ U4 we have two sub cases: (1) x ∉ U3 or (2) x ∈ U3 and x ∈ U4. Suppose (1) x ∉ U3, then (a) y ∉ U1 ⇒ x ∉ U1 \ U3 and y ∈ U3 \ U1 and (U1 \ U3) ∩ (U3 \ U1) = ∅. (b) y ∈ U1 and y ∈ U3. We have x ∈ U1 \ U3, y ∈ U3. Therefore (U1 \ U3) ∩ U3 = ∅. Suppose (2) x ∈ U3, and x ∈ U4, then y ∈ U3 \ U4, x ∈ U4. Therefore (U3 \ U4) ∩ U4 = ∅. This proves that (X, τ) is pgpr-D2.

**CONCLUSION**

We studied pgpr-Di spaces (i = 0, 1, 2) and proved that every topological space is pgpr-D0. We also established that the separation axiom pgpr-D1 is equivalent to pgpr-D2 and every topological space is pgpr-D1 (i = 0, 1, 2).

**REFERENCES**