ON H-PROJECTIVE TRANSFORMATIONS IN ALMOST KAHLERIAN SPACES

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ABSTRACT

Otaba (1956) has studied affine transformation in an almost complex manifold with a natural connection. Ishihara (1957) has defined and studied the Holomorphically projective changes and there groups in an almost complex manifold. Further, Sumitomo (1959) has defined and studied on a holomorphically projective correspondence in an almost complex space. Singh and Samyal (2004) have defined and studied on a Tachibana space with parallel Bochner curvature tensor.

In the present paper, we have defined and studied on holomorphically projective transformations in Almost Kaehlerian spaces and several theorems have been established.


INTRODUCTION

An almost Kaehlerian space is first of all an almost complex space, that is, a 2n-dimensional space with an almost complex structure $F_h$:

$$F_i F_h = \delta_h F_i$$  (1.1)

And always admits a positive definite Riemannian metric tensor $g_{ij}$ satisfying:

$$F_i F_h g_{hi} = g_{ii}$$  (1.2)

From which $F_i = \delta_h F_{ij}$  (1.3)

Where $F_{ij} = F_{ji} g_{ij}$  (1.4)

And finally has the property that the differential form $F_{ij} d\alpha_i \wedge d\alpha_j$ is closed, that is,

$$F_{ijh} = \nabla F_{ih} = \nabla F_{hi} + \nabla h F_{ij} = 0$$  (1.5)

From which $\nabla F_{ij} = 0$  (1.6)

And

$$F_i = -\nabla F_{ij} = 0$$  (1.7)

Here $\nabla$ denotes the operation of covariant differentiation with respect to the Riemannian connection $\{\alpha_i\}$. The Nijenhuis tensor $N_{\alpha h}$ is written in the form:

$$N_{\alpha h} = -4 ( \nabla F_i F_h + 2 G_{ji} F_h + F_i G_{hi} - F_h G_{ji} )$$  (1.8)

A contravariant almost analytic vector field is defined as a vector field $V_i$, satisfying Tachibana (1959):

$$\nabla V_j - \nabla V_j = \alpha_i ; \nabla \{\alpha_i\} - \alpha_i \nabla \{\alpha_i\}$$  (1.19)

Where $\nabla \{\alpha_i\}$ stands for the Lie-derivative with respect to $V_i$. Let $R_{\alpha h}$ be the Riemannian curvature tensor and put

$$R_{\alpha h} = R_{ijh}, \quad R_{ijh} = R_{ijh} g_{hi}, \quad R = R_i F_i$$

Then the following identities are satisfied (Yano 1957)

$$R_{\alpha iji} F_{hj} = R_{hji} F_{ih}, \quad R_{\alpha iji} F_{hj} = R_{hji} F_{ih}$$  (1.9)

$$R_{ijh} = R_{ijh} F_{ih} + R_{ih} F_j = 0$$  (1.10)

$$S_{ij} + S_{ji} = 0, \quad S_{ij} = S_{ij} F_{h}, \quad S_{ji} = -\frac{1}{2} F_{ij} R_{ijh}$$  (1.11)

The holomorphically projective curvature tensor $P_{\alpha iji}$, which will be briefly called HP-curvature tensor, is given by

$$P_{\alpha iji} F_{h} = \frac{1}{2} \left( R_{ij} \delta_{\alpha h} - R_{ij} \delta_{\alpha h} + S_{ij} F_{h} - S_{ij} F_{h} + 2 S_{ij} F_{h} \right)$$  (1.12)

We can obtain the following identities

$$P_{\alpha iji} F_{h} = 0, \quad P_{\alpha iji} F_{h} = 0$$  (1.13)

$$P_{ij} = 0$$  (1.14)

$$P_{\alpha iji} F_{h} = P_{\alpha iji} F_{h} ; P_{\alpha iji} F_{h} = P_{\alpha iji} F_{h}$$  (1.15)

From which, we have

$$P_{\alpha iji} F_{h} = 0$$  (1.16)

$$P_{ij} F_{i} = 0$$  (1.17)

A necessary and sufficient condition for $P_{\alpha iji} F_{h} = 0$, is that the space is a space of constant holomorphically curvature (Tashiro 1957), i.e., a space whose curvature tensor $R_{\alpha h}$ takes the form

$$R_{\alpha h} = -\frac{R}{n+2} \left( G_{hi} \delta_{\alpha h} - G_{hi} \delta_{\alpha h} + F_{hi} F_{h} - F_{hi} F_{h} + 2 F_{hi} F_{h} \right)$$  (1.18)

For a vector field $V$ and a tensor field $\alpha_{\alpha i}$ the following identities are known (Yano 1957)

$$\nabla V_j \alpha_{\alpha h} - \nabla V_j \alpha_{\alpha h} = \alpha_i ; \nabla \{\alpha_i\} - \alpha_i \nabla \{\alpha_i\}$$  (1.19)

$$\nabla V_j \alpha_{\alpha h} - \nabla V_j \alpha_{\alpha h} = \alpha_i ; \nabla \{\alpha_i\} - \alpha_i \nabla \{\alpha_i\}$$  (1.20)

Where $\nabla \{\alpha_i\}$ denotes the operator of Lie-differentiation with respect to $V_i$.

A Killing vector or an infinitesimal isometry $V$ is defined by

$$E_{\alpha} g_{\alpha} = \nabla_j V_i + \nabla_j V_i = 0$$
Here we shall identify a contravariant vectors $V^i$ with a covariant vector $V_i = g_{ij} V^j$. Hence we shall say $V_i$ is a Killing vector, or that $\rho^i$ is gradient, for example.

An infinitesimal affine transformation $V$ is defined by

$$E_{\rho}(\{h\}) = \nabla V, \nabla V = R_{\rho}$$

$$E_{\rho}(\{h\}) = \rho_j^i \left[ \delta^i_j + \frac{\partial}{\partial x^j} \frac{\partial x^i}{\partial x^j} \right] = R_{\rho}$$

Where $\rho_j^i$ is a certain vector and $\bar{\rho}_j^i = F_i^j \rho_j^i$. In this case, we shall called $\rho_j^i$ the associated vector of the transformation, If $\rho_j^i$ vanishes, then the HP-transformation reduces to an affine one.

Contracting the last equation with respect to $h$ and $i$, we get

$$V^j \frac{\partial}{\partial x^j} = (\rho^i + \delta)^i$$

Where $\Delta^i$ shows that the associated vector is gradient.

A vector field $V$ is called contravariant analytic or, for simplicity, analytic, if it satisfies

$$E_x F_{\rho} \equiv - F_i^j \nabla V + F_i^j \nabla V = 0.$$

### A Geometrical Interpretation of an Analytic HP Transformation

In a differentiable space $M$, we consider a tensor valued function $V$ depending not only on a point $P$ of $M$ but also on $k$ vectors $u_i, u_2, \ldots, u_k$ at the point and denote by $V(p, u_i, u_2, \ldots, u_k)$. We assume that the value of this function $V$ lies in the tensor space associated to the tangent space of $M$ at $P$ and that it depends differentially on its arguments.

Assuming the space $M$ to be affinely connected, we take an arbitrary curve $C$: $x^i = x^i(t)$ and denote its successive derivatives by

$$\frac{dx^i}{dt}, \frac{d^2 x^i}{dt^2}, \ldots, \frac{d^n x^i}{dt^n}$$

Then if we substitute (2.1) into the function $V$ instead of $u_1$, $u_2, \ldots, u_k$ we have a family of tensors $V(C) = V(\bar{x}^i(t) + \Delta x^i(t))$ along the curve $C$.

Let $V$ be an infinitesimal transformation, i.e., a vector field, and $x^i = x^i(t)$ be the infinitesimal point transformation determined by $V$, being an infinitesimal constant. Given a curve $C: x^i = x^i(t)$, the image $C$ of $F$ is expressed by

$$x^i = x^i(t) + \epsilon Z(x)(t),$$

We shall call the limiting value

$$E_{\epsilon}(V(C) \equiv \lim_{\epsilon \to 0} \frac{V(C) - V(C)}{\epsilon}$$

The Lie-derivative of $V(C)$ with respect to $V$, where we have denoted by $V(C)$ the family of tensors induced by $V$ through the transformation $x^i = x^i + \epsilon V^i$.

In a Almost Kaehlerian space, a curve $x^i = x^i(t)$ defined by

$$\frac{d^2 x^i}{dt^2} + \{ h^i \} \frac{d^2 x^i}{dt^2} = \alpha \frac{d^2 x^i}{dt^2} + \beta F_j^i \frac{d^2 x^i}{dt^2}$$

is, by definition, a holomorphically planar curve, or an H-plane curve, where $\alpha$ and $\beta$ are certain functions of $t$.

Let $V$ be an infinitesimal transformation and assume that any $\epsilon$ the infinitesimal point transformation $x^i = x^i(t)$ maps any H-plane curves.

Now we ask for the condition that $V$ preserve that H-plane curves. For such a vector $V$ taking account of (2.2), we have

$$E_{\epsilon}(\frac{d^2 x^i}{dt^2} + \{ h^i \} \frac{d^2 x^i}{dt^2} = \alpha \frac{d^2 x^i}{dt^2} + \beta F_j^i \frac{d^2 x^i}{dt^2}$$

along any H-plane curve, where $\gamma$ and $\delta$ are certain functions of $t$.

Denoting the Lie-derivative of the Christoffel’s symbols and the complex structure $F^h$, respectively, by

$$t^h_i = E_{\epsilon}(\{ h^i \}), \quad a^h_i = E_{\epsilon} F^h_i,$$

We have from (2.3)

$$t^h_i = \gamma^i + \alpha \chi^i + b F^h_i \chi^i - \beta \alpha^h_i \chi^i = 0$$

(2.4)

Where we have put

$$a = -(\gamma + E_{\epsilon} \alpha), b = (\delta + E_{\epsilon} \beta), \quad \chi = \frac{d^2 x^i}{dt^2}$$

Since the relation (2.4) holds for any H-plane curve C, it must hold identically for any values of $\chi$ and $\chi^i$.

By means of the definition of the H-plane curve, we see further that the identity (2.4) holds for any value of the coefficient $\beta$.

Taking account of these arguments, we can easily see that relation

$$a^h_i = f \chi^i + g F^h_i \chi^i$$

hold for any values $\chi^i$ and $\chi^i$, where $f$ is $g$ for $q$ and are certain functions of $\chi$ and $\chi^i$.

Let $\alpha$ be a tensor on $V$ such that

$$F_t^i \alpha^i + a^i \alpha^i = 0.$$ We obtain by means of (2.5)

$$a^h_i \equiv E_{\epsilon} F^h_i = 0.$$ (2.7)

On the other hand, If we substitute (2.7) and $F_t^i = 0$ into the identity

$$V E_{\epsilon} F^h_i - F_t^i E_{\epsilon} V = F_t^i F_t^j \{ \epsilon^i \} - F_t^i F_t^j E_{\epsilon} \{ \epsilon^h_i \}$$

Then we get

$$t^i_t^j F_t^i = t^h_i F_t^i,$$ (2.8)

From (2.6) and (2.8), taking account of the fact that

$$t^i_t^j = a_i \delta^i + \alpha_i \delta^i - \bar{a}_i \delta^i - \bar{a}_i \delta^i$$

Where $\alpha_i$ is certain and vector and $\bar{a}_i = F_t^i \alpha$, we get

$$t^h_i = E_{\epsilon} \{ \epsilon^h_i \} = F_t^i \delta^i + m^i \delta^i - \bar{m} \delta^i - \bar{m} \delta^i,$$ (2.9)

Where $\rho_i$ is a certain vector field. Therefore, the infinitesimal transformation $V$ is an analytic HP-transformation.

Conversely, it is obvious that an analytic HP-transformation preserves the H-plane curves.

Thus we have the following:

**THEOREM (2.1):** In an almost Kaehlerian space, an infinitesimal transformation preserves the H-plane curves, and only if it is also an analytic HP-transformation.

### SOME PROPERTIES OF HP-TRANSFORMATIONS

Let $V$ be an HP-transformation, then it holds

$$E_{\epsilon} \{ \epsilon^h_i \} \equiv \nabla \nabla V^i + R_{\rho}^i \nabla V^i = \rho_i \delta^i + \rho_i \delta^i - \bar{m} \delta^i - \bar{m} \delta^i$$

(3.1)

Transvecting (3.1) with $g^i$, we have

$$\nabla \nabla V^i + R_{\rho}^i \nabla V^i = 0.$$ (3.2)

Hence, by virtue of the well-known theorem on analytic vectors, Yano (1957), Lichnerowicz (1957), we have the following:

**THEOREM (3.1):** In a compact almost Kaehlerian space an HP-transformation is analytic.

In a compact almost Kaehlerian space, $M$, it holds that

$$\int M R_{\rho}(\nabla \nabla V^i) \geq 0.$$

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For an analytic vector $V^i$, where $d\sigma$ denote the volume element of M and the equality holds when and when only $V^i$ is parallel. Therefore, if the Ricci’s from $R_{iji}$ is negative definite, then there exists no non-trivial HP-transformation provided that the space is compact. Taking account of the identity (1.19), we have for a vector field $V^i$

$$\nabla_i F^h_j - \nabla_j F^h_i = F^t_i \nabla_j \{ r, i \} - F^t_j \nabla_i \{ r, j \}$$,

Which implies

$$\nabla_i F^h_j = F^t_i \nabla_j \{ r, i \} - F^t_j \nabla_i \{ r, j \}$$.

Because of $\nabla_i F^h_j = 0$. If the vector field $V^i$ is an HP-transformation, it is easily verified that the right hand-side of the last equation vanishes. Thus we have the following theorems by the virtue of Obata’s theorem Obata (1956):

**THEOREM (3.2):** In an irreducible almost Kaehlerian space admitting no quaternion structure, any HP-transformation is analytic.

**THEOREM (3.3):** In an irreducible almost Kaehlerian space having non-vanishing Ricci tensor any HP-transformation is analytic.

**COROLLARY (3.1):** In an irreducible almost Kaehlerian Einstein space if its scalar curvature is non-vanishing, any HP-transformation is analytic.

In the following part of this section, we shall give some formulae on analytic HP-transformation which will be useful in the further study. Let $V^i$ be an HP-transformation. Substituting (3.1) into the identity

$$\nabla_i \nabla_j g_{ji} - \nabla_j \nabla_k g_{ji} = g_{ji} \nabla_k (k, i) + g_{ji} \nabla_k (k, i)$$,

We find

$$\nabla_i \nabla_j g_{ji} = \rho_j g_{ki} + \rho_k g_{ij} - \rho_i F_{ki} - \rho_i F_{kj} + 2 \rho_k g_{ji}$$.

(3.3)

If we substitute (3.1) into (1.20), then we have

$$\nabla_i \nabla_j g_{ji} = \delta_j \nabla_k \rho_i - \delta_i \nabla_k \rho_j - F^h_i \nabla_k \rho_i$$

$$+ F^h_j \nabla_k \rho_i - (\nabla_k \rho_i - \nabla_l \rho_j) F^h_l$$,

(3.4)

Contracting the last equation with respect to $h$ and $k$ we have

$$\nabla_i \nabla_j g_{ji} = -n \nabla_i \rho_j - 2 F^h_i F^t_j \nabla_t \rho_t$$

(3.5)

Now we shall assume that $V^i$ is an analytic HP-transformation. Then we have

$$\nabla_i \nabla_j g_{ji} = \nabla_i \nabla_j g_{ji}$$

By virtue of (2.1). Hence from (3.5) it follows

$$\nabla_i \rho_j = - F^h_i F^t_j \nabla_t \rho_t$$.

(3.6)

Since $n > 2$. The last equation also is written in the form:

$$\nabla_i \rho_j = - F^h_i F^t_j \nabla_t \rho_t + F^h_j \nabla_i \rho_t = 0$$,

Which shows that $\rho^i$ is analytic. Moreover, according to (3.6) we have

$$\nabla_i \rho_j + \nabla_j \rho_i = F^h_i (\nabla_t \rho_t - F^t_j F^h_j \nabla_t \rho_j) = 0$$.

(3.7)

Which means that $\rho^i$ is a Killing vector. Thus we get the following:

**THEOREM (3.5):** If a vector $\rho^i$ is the associated vector of an analytic HP-transformation, then $\rho^i$ is analytic and $\rho^i$ is a Killing vector.

Now, from (3.5) and (3.6) it follows

$$\nabla_i \rho_j = -(n + 2) \nabla_j \rho_i$$.

(3.8)

From which we have

$$\nabla_i S_{ji} = (n + 2) \nabla_j \rho_i$$.

(3.9)

On the other hand, from (3.4) and (3.7) we get

$$\nabla_i \nabla_j \rho_{ji} = \delta_j \nabla_k \rho_i - \delta_i \nabla_k \rho_j - F^h_i \nabla_k \rho_i + F^h_j \nabla_k \rho_i - 2 F^h_k \nabla_k \rho_i$$.

(3.10)

If we substitute (3.8) and (3.9) into (3.10). Then we can verify Ishihara (1957)

$$\nabla_i \rho_j = \frac{1}{n + 2} \nabla_j \rho_i$$.

(3.11)

In the next place, substitute (3.1) and (3.8) into the identity

$$\nabla_i \nabla_j \rho_{ji} - \nabla_j \nabla_i \rho_{ji} = -R_{ji} \nabla_i (k, r) + R_{ji} \nabla_i (k, r)$$,

We have

$$\nabla_i \nabla_j \rho_{ji} = -(n + 2) \nabla_i \rho_i - \rho_i \rho_i - \rho_i \rho_i + S_{ji} \rho_i + S_{ji} \rho_i + 2 \rho_i \rho_i$$.

(3.12)

Hence we put

$$\rho_{ji} = \frac{1}{n + 2} \nabla_i \rho_i$$.

(3.13)

It holds

$$\nabla_i \rho_{ji} = \rho_i$$.

(3.14)

4. AN ANALYTIC HP-TRANSFORMATION WHICH LEAVES INVARIANT THE COVARIANT DERIVATIVE OF THE HP-CURVATURE TENSOR.

In this section, we shall show an analogous theorem to the one obtained by T. Sumitomo for an infinitesimal projective transformation in a Riemannian space Yano and Nagano (1957). Sumitomo (1959).

Let $V^i$ be an analytic HP-transformation. If we substitute (3.1) and (3.11) into the identity

$$\nabla_i \nabla_j \rho_{ji} - \nabla_j \nabla_i \rho_{ji} = \rho_{ji} \nabla_i (k, r) + \rho_{ji} \nabla_i (k, r)$$,

Then we obtain

$$\nabla_i \nabla_j \rho_{ji} = T_{ji}$$,

Where we have put

$$T_{ji} = \delta_i \nabla_j \rho_{ji} - 2 \rho_j \rho_{ji} - \rho_i \rho_{ji} - \rho_i \rho_{ji} - \nabla_j \rho_{ji} - F^h_i \nabla_j \rho_{ji}$$.

(4.1)

Contracting this equation with respect to $h$ and $i$, we can verify

$$\rho_{ji} = 0$$,

By virtue of (1.13) ~ (1.17).

Substituting the last equation into (4.1) and taking account of $P_{ji} \rho_i$, we obtain the equation

$$2 \rho_i \rho_i + \rho_j \rho_{ji} + \rho_i \rho_{ji} + \rho_i \rho_{ji} = F^h_i \rho_{ji} + \rho_j \rho_{ji} + \rho_i \rho_{ji}$$.

Transvecting this equation with

$$\rho_i \rho_{ji} = \rho_j \rho_{ji}$$.

And taking account of (1.13) ~ (1.17), we obtain

$$\rho_{ji} = \rho_i \rho_{ji} + 2 \rho_i \rho_{ji} + \rho_i \rho_{ji}$$.

After some complicated calculation.

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Since the each term in the left hand side of the last equation is non-negative, it must hold $\rho_I P_{kji}^h = 0$, from which we get the following:

**Theorem (4.1):** If an almost Kaehlerian space admits and analytic non-affine HP-transformation which leaves invariant the covariant derivative of the curvature tensor, then the space is a space of constant holomorphic curvature.

In a symmetric almost Kaehlerian space, i.e., in a Kaehlerian space satisfying $\nabla_{\nabla} P_{kji}^h = 0$, the equation $\nabla_{\nabla} P_{kji}^h = 0$ trivially holds, so we have

**Corollary (4.1):** If a symmetric almost Kaehlerian space admits an analytic non-affine HP-transformation, then the space is a space of constant holomorphic curvature.

**References**


