EXPLICIT ANALYTICAL SOLUTIONS FOR THE WAVE-LIKE EQUATIONS IN (1+1)-, (2+1)- AND (3+1) - DIMENSIONS

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1. INTRODUCTION.
The Adomian decomposition method was introduced and developed by George Adomian in [1-2]. A considerable amount of research work has been invested recently in applying this method to a wide class of algebraic, differential, integro-differential, differential-delay and partial differential equations. The method has proved to be powerful and effective where it demonstrates fast convergence of the solution and therefore provides several significant advantages.

This paper is organized as follows:
In Section 2, we write the method of solution.
In Section 3, we give some illustrative examples that clarify the application of the Adomian decomposition method in distinct dimensions mentioned above to the wave-like equations in homogeneous and inhomogeneous cases.
In Section 4, we give some conclusions.

2. Method of solution[9-14]
The Adomian decomposition method will be illustrated by discussing the following typical wave model.

Without loss of generality, as a simple wave equation, we consider the following initial-boundary value problem:
The partial differential equation (PDE)

\[ u_{tt} = c^2 u_{xx} \, , \quad 0 < x < L, \, t > 0, \]  

(1)

, with boundary conditions (BC)

\[ u(0,t) = 0, u(L,t) = 0, t \geq 0, \]  

(2)

, and the initial conditions (IC)

\[ u(x,0) = f(x), u_t(x,0) = g(x), \]  

(3)

, where \( u = u(x,t) \) is the displacement of any point of the string at the position \( x \) and time \( t \), and \( c \) is a constant related to the elasticity of the material of the string.

In an operator form, Eq.(1) can be written as

\[ L_t u(x,t) = c^2 L_x u(x,t) \]  

(4)
Following the analysis discussed above, we set the recursive relation

\[
\sum_{n=0}^{\infty} u_n(x,t) = f(x) + t g(x) + c^2 L_t^{-1}(L_x^{-1} u(x,t)).
\]

(9)

Substituting (8) into (7) yields

\[
\sum_{n=0}^{\infty} u_n(x,t) = f(x) + t g(x) + c^2 L_t^{-1}(L_x^{-1} (\sum_{n=0}^{\infty} u_n(x,t)));
\]

Following the Adomian analysis, we set the recursive relation

\[
u_0(x,t) = f(x) + t g(x) ,
\]

\[
u_{k+1}(x,t) = c^2 L_t^{-1}(L_x^{-1} (\sum_{k=0}^{\infty} u_k(x,t))), \quad k \geq 0.
\]

Having determined the components \(u_n(x,t)\), the series solution of \(u(x,t)\) is readily obtained. In this work, the serious solution will be used to provide the solution in a closed form.

### 3. Some Illustrative examples.

To incorporate our discussion, three important cases of wave equations with specific initial-boundary conditions, will be investigated to show the reliability of the proposed scheme [9-14].

#### 3.1. One Dimensional Wave-like Equation (i.e in 1+1-Dimensions)

**Example 1. (Homogeneous Wave Equation)**

Consider the initial-boundary value problem

**PDE**

\[
u_{tt} = u_{xx} , \quad 0 < x < \pi , \quad t > 0,
\]

(11)

**BC**

\[
u(0,t) = 1, \quad u(\pi,t) = 1 + \pi , \quad t \geq 0,
\]

\[
u(x,0) = 1 + x , \quad \nu_t(x,0) = \sin x,
\]

(12) IC

(13)

Operating with \(L_t^{-1}\) on both sides of (11) gives

\[
u(x,t) = 1 + x + t \sin x + L_t^{-1}(L_x^{-1} u(x,t)).
\]

(14)

Substituting the series of components (8) for \(u(x,t)\) into (14), we obtain

\[
\sum_{n=0}^{\infty} u_n(x,t) = 1 + x + t \sin x + L_t^{-1}(L_x^{-1} (\sum_{n=0}^{\infty} u_n(x,t))).
\]

(15)

Following the analysis discussed above, we set the recursive relation
\[ u_0(x,t) = 1 + x + t \sin x, \quad (16) \]
\[ u_k + 1(x,t) = L_t^{-1}(L_x(u_k)), \quad k \geq 0, \quad (17) \]
for the determination of the components \( u_n(x,t) \) of the solution \( u(x,t) \).

The first few terms of the decomposition (8) are given by
\[ u_0(x,t) = 1 + x + t \sin x, \]
\[ u_1(x,t) = - \sin x \left( \frac{t^3}{3!} \right), \]
\[ u_2(x,t) = \sin x \left( \frac{t^5}{5!} \right), \]
\[ u_3(x,t) = - \sin x \left( \frac{t^7}{7!} \right), \quad (18) \]
... and so on. Combining (8) and (18), the solution in a series form is given by
\[ u(x,t) = 1 + x + \sin x(t - \frac{t^3}{3!} \frac{t^5}{5!} \frac{t^7}{7!} + ...) \quad (19) \]
and the exact (closed form) solution is
\[ u(x,t) = 1 + x + \sin x \cdot \sin t \quad (20) \]

**Example 2. (Inhomogeneous Wave-Like Equation)**

Consider the initial-boundary value problem
\[ \text{PDE} \quad u_{tt} = u_{xx} - 3u + 3, \quad 0 < x < \pi, \quad t > 0, \quad (21) \]
\[ \text{BC} \quad u(0,t) = 1, \quad u(\pi,t) = 1, \quad t \geq 0, \quad (22) \]
\[ \text{IC} \quad u(x,0) = 1, u_t(x,0) = 2 \sin x, \quad (23) \]

Operating with \( L_t^{-1} \) on both sides of (21) gives
\[ u(x,t) = 1 + \frac{3t^2}{2} + 2t \sin x + L_t^{-1}(L_x(u(x,t)) - 3u(x,t)). \quad (24) \]

Substituting the series of components (8) for \( u(x,t) \) into (24), we obtain
\[ \sum_{n=0}^{\infty} u_n(x,t) = 1 + \frac{3t^2}{2} + 2t \sin x + L_t^{-1}(L_x(\sum_{n=0}^{\infty} u_n(x,t)) - 3 \sum_{n=0}^{\infty} u_n(x,t)). \quad (25) \]
Following the analysis discussed above, we set the recursive relation
\[ u_0(x,t) = 1 + \frac{3t^2}{2} + 2t \sin x, \quad (26) \]
\[ u_k + 1(x,t) = L_t^{-1}(L_x(u_k)) - 3u_k, \quad k \geq 0, \quad (27) \]
for the determination of the components \( u_n(x,t) \) of the solution \( u(x,t) \).

The first few terms of the decomposition (8) are given by
\[ u_0(x,t) = 1 + \frac{3t^2}{2} + 2t \sin x, \]
\[ u_1(x,t) = - \left( \frac{(2t)^3}{3!} \right) \sin x - \frac{3t^2}{2} - \frac{9t^4}{4!}, \]
\[ u_2(x,t) = \frac{4(2t)^5}{5!} \sin x + \frac{(3t^4)^2}{4!} + \frac{3(3t^6)^2}{6!} \quad (28) \]
... and so on.

Combining (8) and (28), the solution in a series form is given by
\[ u(x,t) = 1 + \sin x \left( \frac{2t}{1!} - \frac{(2t)^3}{3!} + \frac{(2t)^5}{5!} - \ldots \right), \quad (29) \]

the exact (closed form) solution is
\[ u(x,t) = 1 + \sin x \sin 2t. \quad (30) \]

### 3.2. Two Dimensional Wave - Like Equation (i.e. in 2+1-Dimensions)

Extending the application of the Adomian decomposition to the two dimensional case in a direct and straightforward fashion, as seen in the following examples:

**Example 3. (Homogeneous Wave - Like Equation)**

Consider the initial-boundary value problem

**PDE** \[ u_{tt} = u_{xx} + u_{yy}, \quad 0 < x, y < \pi, \ t > 0, \quad (31) \]

**BC**
\[ u(0,y,t) = 1 + \sin y \sin t, \quad u(x,\pi, t) = 1 + \pi + \sin y \sin t, \quad \]
\[ u(x,0,t) = u(x,\pi, t) = 1 + x, \quad \]
\[ u(x,y,0) = 1 + x, \quad u_t (x,y,0) = \sin y, \quad (32) \]

Operating with \( L_t^{-1} \) on both sides of \( (31) \) gives
\[ u(x,y,t) = 1 + x + t \sin y + L_t^{-1}(L_x u(x,y,t) + L_y u(x,y,t)). \quad (34) \]

Following the analysis discussed above, we set the recursive relation
\[ u_0(x,y,t) = 1 + x + t \sin y, \quad (35) \]
\[ u_{k+1}(x,y,t) = L_t^{-1}(L_x (u_k) + L_y (u_k)), \quad k \geq 0, \quad (36) \]

determination of the components \( u_n(x,y,t) \) of the solution \( u(x,y,t) \).

The first few terms of the decomposition \( (8) \) are given by
\[ u_0(x,y,t) = 1 + x + t \sin y, \]
\[ u_1(x,y,t) = -\frac{(t)^3}{3!} \sin y, \]
\[ u_2(x,y,t) = \frac{(t)^5}{5!} \sin y, \quad (37) \]
\[ \ldots \]

Therefore the exact (closed form) solution is
\[ u(x,y,t) = 1 + x + \sin y \sin t. \quad (38) \]

**Example 4. (Inhomogeneous Wave Equation)**

Consider the initial-boundary value problem

**PDE** \[ u_{tt} = u_{xx} + u_{yy} - 4, \quad 0 < x, y < \pi, \ t > 0, \quad (39) \]

**BC**
\[ u(0,y,t) = y^2, \quad u(\pi, y, t) = \pi^2 + y^2 \]
\[ , u(x,0,t) = x^2 + \sin x \sin t, \quad u(x,\pi, t) = \pi^2 + x^2 + \sin x \sin t \]

**IC**
\[ u(x,y,0) = x^2 + y^2, \quad u_t (x,y,0) = \sin x. \quad (41) \]

Operating with \( L_t^{-1} \) on both sides of \( (39) \) gives
\[ u(x,y,t) = x^2 + y^2 - 2 t^2 + t \sin x + L_t^{-1}(L_x u(x,y,t) + L_y u(x,y,t)). \quad (42) \]

Following the analysis discussed above, we set the recursive relation
\[ u_0(x,y,t) = x^2 + y^2 - 2 t^2 + t \sin x, \quad (43) \]
\[ u_{k+1}(x, y, t) = L^{-1}_L \left( L_x (u_k) + L_y (u_k) \right), \quad k \geq 0, \]  
\[ , \) \]

For the determination of the components \( u_n(x, y, t) \) of the solution \( u(x, y, t) \).

Following the same procedures as in the above examples, we obtain the exact solution as

\[ u(x, y, t) = x^2 + y^2 + \sin x \sin t. \]  

3.3 Three Dimensional Wave-Like Equation (i.e. in 3+1-Dimensions)

Also the extension is continued to the three-dimensional case, in the following examples:

**Example 5. (Homogeneous Wave-Like Equation)**

Consider the initial-boundary value problem

**PDE**

\[ u_{tt} - u_{xx} + u_{yy} + u_{zz} - 6u, \quad 0 < x, y, z < \pi, \quad t > 0, \]

\[ u(0, y, z, t) = u(\pi, y, z, t) = 0, \]

\[ u(x, 0, z, t) = u(x, \pi, z, t) = 0, \]

\[ u(x, y, 0, t) = u(x, y, \pi, t) = 0 \]

\[(46) \text{ BC} \]

**IC**

\[ u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = 3 \sin x \sin y \sin z. \]

\[(47) \]

Operating with \( L_t^{-1} \) on both sides of (46) gives

\[ u(x, y, z, t) = 3t \sin x \sin y \sin z + L_t^{-1} (L_x u(x, y, z, t) + L_y u(x, y, z, t)) \]

\[ + L_z u(x, y, z, t) - 6u(x, y, z, t). \]

\[(49) \]

As in the previous examples, hence we get the exact solution in the following closed form

\[ u(x, y, z, t) = \sin x \sin y \sin z \sin 3t \]

\[(50) \]

**Example 6. (Inhomogeneous Wave Equation)**

Consider the initial-boundary value problem

**PDE**

\[ u_{tt} - u_{xx} + u_{yy} + u_{zz} + \sin x + \sin y, \quad 0 < x, y, z < \pi, \quad t > 0, \]

\[ u(0, y, z, t) = u(\pi, y, z, t) = \sin y + \sin z \sin t, \]

\[ u(x, 0, z, t) = u(x, \pi, z, t) = \sin x + \sin z \sin t, \]

\[ u(x, y, 0, t) = u(x, y, \pi, t) = \sin x + \sin y \]

\[(51) \text{ BC} \]

**IC**

\[ u(x, y, z, 0) = \sin x + \sin y, \quad u_t(x, y, z, 0) = \sin z. \]

\[(52) \]

Operating with \( L_t^{-1} \) on both sides of (51) gives

\[ u(x, y, z, t) = (\sin x + \sin y) \left(1 + \frac{t^2}{2}\right) + t \sin z + L_t^{-1} (L_x u(x, y, z, t) + \]

\[ + L_y u(x, y, z, t) + L_z u(x, y, z, t)). \]

\[(54) \]

As in the previous examples, hence we get the exact solution in the following closed form

\[ u(x, y, z, t) = \sin x + \sin y + \sin z \sin t \]

\[(55) \]

**CONCLUSION.**

We have applied successfully the Adomian decomposition method to wave-like equations in distinct dimensions, i.e. in \((1+1)\), \((2+1)\) and \((3+1)\) dimensions in both homogeneous and inhomogeneous cases. As a result, the exact solutions for
these equations are obtained in a closed form. The method is powerful and effective and can be applied to other equations and this will be done elsewhere.

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