FUZZY MULTIPLE ATTRIBUTE DECISION MAKING BASED ON CONSISTENCY APPROXIMATION AND RANKING METHOD

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INTRODUCTION

Multi-attribute decision-making (MADM) addresses the problem of choosing an optimum choice that has been highest degree of satisfaction from a set of alternatives that are characterized in terms of their attributes. The analytic hierarchy process (AHP) is one of the extensively used multi-criteria decision making methods. However, the conventional AHP still cannot process imprecise or vague knowledge (Wand and Chen, 2008). Fuzzy AHP extension of AHP was developed to solve imprecise hierarchical problems. As part of AHP procedure, a consistency check is required to identify inconsistency matrix. The lack of consistency in decision making can lead to inconsistent conclusions. In fuzzy AHP method, it is difficult to ensure a consistent pair-wise comparison (Wand and Chen, 2008). Since Saaty (1980) put forward the concept of consistency and judged the decision-maker’s judgment matrix by its consistency ratio, there many studies about the ordinal consistency of judgment matrix (Lamata and Pelaez, 2003; Li and Ma, 2003; Alonso and Lamata, 2006). On the basis of AHP consistent judgment matrix, Wang and Guo (2010) directly determine the ideal priority vector by a general formula for solving the fuzzy judgment matrix priority. Wu (2010) used utilizing the continuous interval argument operator and the expected function for ranking, which is able to be adjusted according to decision-maker’s optimistic degree and will be more reasonable.

The purpose of this research is to establish a linear goal programming model and analysis for the parameters in the transformation formulas of fuzzy consistent judgment matrix between the complementary matrix and the appraisal values of all alternatives. Based on this model, we then have developed an approach to solve fuzzy numbers multi-attribute decision making problems. It is illustrated that the ranking formula according to the relation between the weight and the elements of fuzzy consistent judgment matrix, enriching the theory and method of fuzzy decision making.

2. Relative works

A matrix whose elements meet the condition $0 \leq a_{ij} \leq 1$ for $\forall i, j = 1, 2, ..., n$ is called a fuzzy judgment matrix. Research on consistency of fuzzy matrix is necessary and instructive. Lamata and Pelaez (2003) defined the consistency index CI of a matrix using the average of the consistency index of the matrix triplets. Li and Ma (2003) developed a model that can assist on marking a consistent decision and used Gower plots to judge the ordinal consistency graphically. Basile and Dapuzzol (2002) used the complete strict simple order to judge the ordinal consistency of judgment matrix. Luo (2004) studied the revising method of judgment matrix and considered that ordinal consistency was the prerequisite of ordinal consistency. Zhu et al (2007) demonstrated that consistent analysis should be based on ordinal consistency. Alonso and Lamata (2006) introduced a statistical criterion for accepting/rejecting the pair-wise reciprocal comparison matrices in AHP. For comparison matrix which fails the consistency test, the decision maker must redo the ratios. Unlike that AHP method, the ratios are point estimates and the comparison ratios in fuzzy AHP method are given by fuzzy numbers. The decision maker to provide the comparison ratios such that the fuzzy numbers include only consistency would be laborious and high unrealistic. The critical void is not only the need to have a consistency test to accept consistent matrix, but also a mechanism to filter out inconsistent information within a consistent matrix (Wang and Chen, 2008). Preference relations are the most
common representation of information used for solving decision making problems due to their effectiveness in modeling processes. These preference relations can be categorized into multiplicative preference relations (Wang and Fan, 2007; Fan et al. 2006a), fuzzy preference relations (Chiclana et al. 2001; Fan et al. 2006b; Ma et al. 2006; Wang and Chen, 2005; Wang and Fan, 2007; Xu, 2004) and linguistic preference relations (Herrera and Herrera-Viedma, 2000; Herrera et al. 2001a; Herrera and Martinez, 2001; Xu, 2005; Xu, 2007). The ranking of the fuzzy AHP method is rather imprecise, and the subjective judgment by perception, evaluation, improvement and selection based on preference of decision-makers has great influence on the fuzzy AHP results. To overcome these problems, several researchers used some methods such as revising method (Zhang et al. 2010), least deviation priority method (Li and Zhang, 2011). Fan and Jiang (2001) in an overview on ranking method of fuzzy judgment matrix proposed a new concept of consistency based on the new consistency approximation and ranking method to avoid misleading conclusions. This characterization simplifies the analysis of consistency among expert opinions. Thus, this study applies Fan and Jiang’s method to enhance the consistency of the fuzzy AHP method. The proposed method yields decision matrices for making pair-wise comparisons using additive reciprocal property and consistency.

3. Preliminary knowledge

Definition 1: Suppose \( A = (a_{ij})_{n \times n} \) be a judgment matrix given by a decision-maker. If \( 0 \leq a_{ij} \leq 1 \) for \( \forall i, j = 1,2,...,n \), then \( A = (a_{ij})_{n \times n} \) is called a fuzzy judgment matrix, where \( a_{ij} \) is the preference degree of \( x_i \) to \( x_j \).

\[
\begin{align*}
a_{ij} = \begin{cases} 
0 & x_i \text{ is strictly inferior to } x_j \\
0,0.5 & x_i \text{ is indifferent to } x_j \\
0.5 & x_i \text{ is inferior to } x_j \\
(0.5,1) & x_i \text{ is superior to } x_j \\
1 & x_i \text{ is strictly superior to } x_j 
\end{cases}
\end{align*}
\]

Fuzzy judgment matrix shows the preference of pairwise comparison on alternatives. If a fuzzy judgment matrix is not ordinal consistent, the fuzzy judgment matrix is not acceptable (Alonso and Lamata, 2006).

Definition 2: Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy judgment matrix. \( A = (a_{ij})_{n \times n} \) is called fuzzy complementary judgment matrix if it follows.

\[
a_{ij} + a_{ji} = 1, \forall i,j = 1,2,...,n
\]

or \( a_{ij} = 0.5 \) for \( \forall i,j = 1,2,...,n \)

Definition 3: Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy complementary matrix. If the following condition is true, \( A = (a_{ij})_{n \times n} \) is ordinal consistent.

\[
a_{ik} > 0.5, a_{kj} > 0.5 \Rightarrow a_{ij} > 0.5
\]

or \( a_{ik} < 0.5, a_{kj} < 0.5 \Rightarrow a_{ij} < 0.5 \)

Definition 4: Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy complementary fuzzy matrix. If the following condition is true, for \( \forall i, j,k = 1,2,...,n \), then \( A = (a_{ij})_{n \times n} \) is consistency of fuzzy complementary judgment matrix.

\[
a_{ij} = a_{ik} - a_{kj} + 0.5
\]

Proposition 1: Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy complementary fuzzy matrix, then

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n^2 / 2
\]

Proposition 2: (Xu, 2001)

Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy complementary fuzzy matrix. When \( r_{ij} = \frac{r_i - r_j}{\alpha} + 0.5 \), the matrix \( R = (r_{ij})_{n \times n} \) is consistency of fuzzy complementary judgment matrix. Where \( \alpha = 2(n-1) \)

\[
r_{ij} = \sum_{j=1}^{n} a_{ij}, i = 1,2,...,n
\]

Proposition 3: Suppose \( A = (a_{ij})_{n \times n} \) be a fuzzy complementary fuzzy matrix. When

\[
r_{ij} = \frac{r_i - r_j}{2(n-1)} + 0.5
\]

Proof:

\[
r_{ij} = \frac{r_i - r_j}{2(n-1)} + 0.5 + \frac{(r_i - r_j) + (r_k - r_j)}{2(n-1)} + 0.5
\]

By Eq. (4), the matrix \( R = (r_{ij})_{n \times n} \) is consistency of fuzzy complementary judgment matrix.

Proposition 4: The matrix \( R = (r_{ij})_{n \times n} \) is consistency of fuzzy complementary judgment matrix. Then the weights of each criterion are:

\[
w_i = \sum_{j=1}^{n} a_{ij} + \frac{n}{2(n-1)}
\]

4. Ranking methods for the fuzzy consistent judgment matrix

4.1 linear goal programming mode

Fan et al. (2006a; 2006b) discussed this method. Suppose alternative \( x_i \) have ranking value \( w_i \) \( \forall i = 1,2,...,n \), and \( w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \) the matrix \( R = (r_{ij})_{n \times n} \) is consistency of fuzzy complementary judgment matrix. Set
$$r_j = (1 + w_j - w_j)/2$$  \hspace{1cm} (9)$$

The matrix $R = (r_j)_{n \times n}$ satisfies Eq. (4). We use $r_j = (1 + w_j - w_j)/2$ approximation to $a_j$ for calculation $w_j$ if $i = 1, \ldots , n$. That is

$$a_j \approx (1 + w_j - w_j)/2 \text{ } \forall i, j, k = 1, \ldots , n; \ i \neq j$$  \hspace{1cm} (10)$$

From Eq. (10), we know that there have same characters between $a_j$ and $(1 + w_j - w_j)/2$, and build the following objective model

$$
\min Z_j = \left| a_j - (1 + w_j - w_j)/2 \right| \quad \forall i, j, k = 1, \ldots , n; \ i \neq j
$$

(11a)

$$\text{s.t.} \sum_{j=1}^{n} w_j = 1$$

(11b)

$$w_j \geq 0 \quad \forall i = 1, \ldots , n$$

(11c)

In the above model, we can write $Z_j = \left| w_j - w_j \right| - (2a_j - 1)$ and the expected value of $Z_j$ approach to 0. \forall i, j = 1, \ldots , n; \ i \neq j$

We transform the Eq. (11a)-(11c) to the following linear goal programming problem.

$$\min z = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} s_{ij} d_{ij} + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} t_{ij} d_{ij}$$

(12a)

$$\text{s.t.} \quad (w_i - w_j) - (2a_j - 1) - d_{ij} + d_{ij} = 0 \quad \forall i, j = 1, \ldots , n; \ i \neq j$$

(12b)

$$\sum_{j=1}^{n} w_j = 1$$

(12c)

$$w_j \geq 0 \quad \forall i = 1, \ldots , n$$

(12d)

$$d_{ij} \geq 0, d_{ij} \geq 0 \quad \forall i, j = 1, \ldots , n; i \neq j$$

(12e)

Where $d_{ij}$ is deviation of variable that the objective function $Z_j$ is lower than expected value 0. $d_{ij}$ is deviation of variable that the objective function $Z_j$ is higher than expected value 0. $s_{ij}$ and $t_{ij}$ are weight coefficient of $d_{ij}$ and $d_{ij}$ respectively.

We see that Eq. (12b) includes the following constraint conditions:

$$(w_i - w_j) - (2a_j - 1) - d_{ij} + d_{ij} = 0 \quad \forall i, j = 1, \ldots , n; \ i < j$$

(13)

$$(w_i - w_j) - (2a_j - 1) - d_{ij} + d_{ij} = 0 \quad \forall i, j = 1, \ldots , n; \ i > j$$

(14)

We actural consider the $n(n-1)/2$ constraint conditions. In addition, we can let Eq. (12a) all of the objective function is fair competition and there is no preference relation.

Therefore, we take $s_{ij} = t_{ij} = 1, \forall i, j = 1, \ldots , n; \ i < j$. The above model can write as:

$$\min z = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} (d_{ij} + d_{ij})$$

(15a)

$$\text{s.t.} \quad w_i - w_j - d_{ij} + d_{ij} = 2a_j - 1 \quad \forall i, j = 1, \ldots , n; \ i < j$$

(15b)

$$\sum_{j=1}^{n} w_j = 1$$

(15c)

$$d_{ij} \geq 0, d_{ij} \geq 0 \quad \forall i, j = 1, \ldots , n; \ i \neq j$$

(15d)

(15e)

Based on the fuzzy judgment matrix $A = (a_j)_{n \times n}$, it established the linear goal programming model. By solving this model, we can get the required ordering vector $w = (w_1, w_2, \ldots , w_n)^T$. We can construct a consistency of fuzzy complementary judgment matrix $R = (r_j)_{n \times n}$. We obtain the ranking vector of alternatives and the consistent fuzzy judgment matrix for approximating to the decision maker’s preference information.

4.2 Ranking method between the weight and the elements of fuzzy consistent judgment matrix

Definition 5: (Fan and Jiang, 2001)

Suppose $A = (a_j)_{n \times n}$ be a fuzzy complementary judgment matrix. The matrix $V = (v_j)_{n \times n}$ is preference matrix. Where

$$v_j = \begin{cases} 
1, & a_j > 0.5 \\
0, & otherwise \forall i, j = 1, \ldots , n 
\end{cases}$$

(16)

From the graph theory, the relationship of matrix $V$ can be implied a directed graph. This diagram has called the preference graph to correspond the matrix. Matrix $V$ is implied a directed graph. This diagram has called the adjacency matrix of the non-cycle of directed graph.

Definition 6: Reachable matrix

Suppose $A = (a_j)_{n \times n}$ be a fuzzy complementary matrix. Its preference matrix is $V = (v_j)_{n \times n}$. The matrix $T = (t_j)_{n \times n}$ is reachable matrix. Where, the matrix $T = V^{(3)} = (v_j)_{n \times n}$ is the three power of preference matrix of $V = (v_j)_{n \times n}$.

Proposition 5: Suppose $A = (a_j)_{n \times n}$ be a fuzzy complementary matrix. Its preference matrix is $V = (v_j)_{n \times n}$ the judgment matrix $A = (a_j)_{n \times n}$ is ordinal consistency if and only if $T = (t_j)_{n \times n}$ is reachable matrix and the elements of diagonal of $T = (t_j)_{n \times n}$ is zero.

For example, the fuzzy complementary matrix is

$$A = \begin{bmatrix} 
0.5 & 0.7 & 0.4 & 0.5 & 0.7 \\
0.4 & 0.5 & 0.7 & 0.5 & 0.8 \\
0.6 & 0.3 & 0.5 & 0.5 & 0.9 \\
0.5 & 0.5 & 0.5 & 0.5 & 0.7 \\
0.3 & 0.2 & 0.1 & 0.3 & 0.5 
\end{bmatrix}$$

(17)

Using the definition 6, the reachable matrix can be obtained and is shown in Eq. (18) as follows:
By proposition 5, we know that the fuzzy complementary matrix is not ordinal consistent.

In decision-making, an expert may give his preference on alternatives using judgment matrix. To judge the ordinal consistency of it, we can use the reachable matrix. The ordinal consistency of fuzzy matrix and finding the ordering vector $w = (w_1, w_2, ..., w_n)^T$ can be determined with the following four steps.

Step 1: Give the preference matrix $V = (v_{ij})_{n \times n}$ of the fuzzy complementary matrix $A = (a_{ij})_{n \times n}$ using formula Eq. (16).

Step 2: Find the reachable matrix $T = (t_{ij})_{n \times n} = v^{(3)} = (v_{ij}^3)_{n \times n}$ according to Proposition 4.

Step 3: If the elements of diagonal of $T = (t_{ij})_{n \times n}$ are zero, the fuzzy complementary matrix $A = (a_{ij})_{n \times n}$ is ordinal consistent; otherwise, the fuzzy complementary matrix is not ordinal consistent, it need be revised by using proposition 2 and proposition 3.

Step 4: Using formula Eq. (8) can be obtained the required ordering vector $w = (w_1, w_2, ..., w_n)^T$ and $\sum_{j=1}^{n} w_j = 1$.

5.1 Linear goal programming model
Suppose expert gives his preference on alternatives $X = \{x_1, x_2, x_3, x_4\}$. The fuzzy complementary matrix from Chiclana et al. (1998) is:

$$A = \begin{bmatrix}
0.5 & 0.1 & 0.6 & 0.7 \\
0.9 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.9 \\
0.3 & 0.6 & 0.1 & 0.5
\end{bmatrix}$$

(19)

Step 1: The linear goal programming model Eq. (19) by proposition 5, we know that the fuzzy complementary matrix is not ordinal consistent. Therefore, we used the linear goal programming model (Eq. (15a) – (15e)) can be obtained and is shown in Eq. (20) as follows:

$$\min Z = d_{12}^* + d_{13}^* + d_{14}^* + d_{23}^* + d_{24}^* + d_{34}^*$$

$$\begin{align*}
& s.t \quad w_1 - w_2 - d_{12}^* + d_{12}^* = -0.8 \\
& w_1 - w_3 - d_{13}^* + d_{13}^* = 0.2 \\
& w_1 - w_4 - d_{14}^* + d_{14}^* = 0.4 \\
& w_2 - w_3 - d_{23}^* + d_{23}^* = 0.6 \\
& w_2 - w_4 - d_{24}^* + d_{24}^* = -0.2 \\
& w_3 - w_4 - d_{34}^* + d_{34}^* = 0.8
\end{align*}$$

(20)

Step 2: The solution of linear goal programming model Using linear goal programming method, we can obtained the solution of model are:

$$d_{12}^* = 0.0, \quad d_{13}^* = 0.0, \quad d_{14}^* = 0.2, \quad d_{23}^* = 0.0, \quad d_{24}^* = 0.0, \quad d_{34}^* = 0.733$$

Step 3: The required ordering vector We can get the required ordering vector $w = (0.2667, 0.6667, 0.0667, 0.0667, 0.0667)^T$.

5.2 Ranking method
The fuzzy complementary matrix from Chiclana et al. (1998) is:

$$A = \begin{bmatrix}
0.5 & 0.1 & 0.6 & 0.7 \\
0.9 & 0.5 & 0.8 & 0.4 \\
0.4 & 0.2 & 0.5 & 0.9 \\
0.3 & 0.6 & 0.1 & 0.5
\end{bmatrix}$$

(22)

Step 1: The preference matrix $V$ using formula Eq. (16) are shown in Eq. (23)

$$V = \begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}$$

(23)

Step 2: Find the reachable matrix $T$ using proposition 4 are shown in Eq. (24)

$$T = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix}$$

(24)

Step 3: The elements of diagonal of $T = (t_{ij})_{n \times n}$ are not zero; the fuzzy complementary matrix $A = (a_{ij})_{n \times n}$ is not ordinal consistency matrix.

Step 4: The ordering vector $w = (w_1, w_2, ..., w_n)^T$ using formula Eq. (8) are shown in Eq. (25)

$$w = (0.2416, 0.3, 0.25, 0.2084)^T.$$  

(25)
The ranking vector of alternatives is $x_2 > x_3 > x_1 > x_4$.

CONCLUSIONS

Two making formulas of the fuzzy consistent judgment matrix are analyzed in this paper, which are linear goal programming mode and the ranking method given an analysis for parameters in the transformation formulas of the fuzzy consistent judgment matrix and presented a formula for priority of fuzzy complementary judgment matrix. The methods are simple, effective and achieve the features, easy to implement in the computer. The conclusion of this paper is helpful to the correct application of the making formula of fuzzy consistent matrix, enriching the theory and method of fuzzy decision making. For the extension of this work, the relation of two ranking method and disadvantages existing in two methods will be discussed.

REFERENCES